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# Equivalence of Connections

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## 0. INTRODUCTION

**0.1.** Connections on vector bundles over smooth manifolds are fundamental in modern differential geometry and their study has a long history. That the equivalence problem for such connections can be solved in terms of local invariants of finite order is stated in many classical works on the subject. Making these statements precise, however, can be both complicated and frustrating. Nevertheless, the effort is worthwhile since the results are important in various applications.

This work is a generalization and simplification of the geometric portion of our paper [CD1] on the equivalence problem. It forms the basis for our subsequent generalization of the operator portion, which has been announced in [CD2], where consideration of commuting $m$-tuples of operators with an open joint spectrum leads in a natural way to vector bundles over a $k$-dimensional complex base. Our previous equivalence results for bundles were only on a 1-dimensional base. The Equivalence Theorem we give here works for all $k$ and the proof is more conceptual. In this new framework it is also possible to see that the theorem is sharp for vector bundles. Sharpness of the results when applied to operators or to holomorphic curves in Grassmannians is still an open question.

We consider $C^\infty$ Hermitian vector bundles with metric-preserving connections over $\Omega$ an open subset of $\mathbb{C}^k$; $\Omega$ in $\mathbb{R}^k$ can be treated similarly. Such bundles arise naturally in studying certain classes of bounded linear operators on a separable Hilbert space, holomorphic maps into complex Grassmann manifolds, and Hermitian holomorphic vector bundles over

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complex manifolds. Two such bundles with connections are *equivalent* if there exists a bundle map between them that is both an isometry and is connection-preserving. A necessary condition for this to occur is that the curvatures corresponding to the two bundles, and their covariant derivatives to all orders, should be simultaneously unitarily equivalent at each point of $\Omega$. The Equivalence Theorem gives a sufficient condition to achieve equivalence, at least locally and off a closed nowhere dense subset in $\Omega$. As examples show, this is the best that can be expected.

EQUIVALENCE THEOREM (cf. Theorem II, 3.10). *Let $E$ and $\tilde{E}$ be $n$-dimensional $C^\infty$ Hermitian vector bundles over $\Omega$ open in $\mathbb{C}^k$, with metric-preserving connections $D$ and $\tilde{D}$, respectively. Let $\Omega_0$ be the open dense subset of $\Omega$ on which the dimension of the algebra generated by the curvatures of $D$ and their covariant derivatives to all orders is locally constant. If at each point $z$ of $\Omega$, the curvatures of $D$ and $\tilde{D}$ and their covariant derivatives to order $n$ are simultaneously unitarily equivalent (where the unitary may depend on $z$) then $E$ and $\tilde{E}$ restricted to $\Omega_0$ are locally equivalent. When $E$ and $D$ are real-analytic, then $\Omega_0$ in fact equals $\Omega$.*

Furthermore, for a generic connection $D$ on $E$, unitary equivalence to order 1 suffices when the base dimension $k$ is greater than 1, and order 2 suffices when $k$ equals 1. There exist (non-generic) examples which show that order $n-1$ does not suffice in general.

Since this paper is directed to an audience with a wide range of backgrounds, we supply more detail than might be usual. Although some individual steps in the development, in particular in Section 1, may seem pedestrian, suppressing them would, we feel, make the paper inaccessible to many readers.

**0.2.** The plan of this paper is as follows:

0. *Introduction.* We give a quick review of the basic notions of Hermitian bundles, connections, equivalence, the canonical connection on a Hermitian holomorphic bundle, curvature, and the like.

1. $C^\infty$ *Block Diagonalization.* This is an expository account of several results on complex matrix algebras (closed under conjugate transpose) extended to the case of $C^\infty$ families of algebras. We show that every $C^\infty$ $*$-algebra of matrices can (locally) be put in the form of a sum of full matrix algebras with multiplicities, what we call a $C^\infty$ block diagonalization, by conjugating with some $C^\infty$ unitary-valued matrix function. This holds as long as the dimension of the algebra is constant. Furthermore, if we have a chain of such $C^\infty$ families of algebras then the diagonalization can be done simultaneously. We also show that any $C^\infty$ $*$-isomorphism of such algebras, subject to some obvious necessary hypotheses, is given by conjugation with a $C^\infty$ unitary.

2. *Diagonalization of Connections.* We describe how diagonalization of the algebra $\mathscr{A}^{\mathscr{K}}(z)$ generated by the curvatures and their covariant derivatives at $z$ leads to a diagonalization of the connection itself.

We give a detailed discussion of what is the minimal order of covariant derivative required to generate $\mathscr{A}^{\mathscr{K}}(z)$ at a generic point $z$. We introduce the notion of the *coalescing set* of $\mathscr{A}^{\mathscr{K}}(z)$, the set where the dimension of $\mathscr{A}^{\mathscr{K}}(z)$ is not locally constant. We show that the minimal order, which we call the *generating order*, is less than $n$, the dimension of the bundle, for $z$ not in the coalescing set. When the connection is real-analytic we show that the coalescing set is empty, which is not necessarily true in general. For a generic connection, the generating order is 1 if the base dimension $k$ is 1 and is zero otherwise. That is, $\mathscr{A}^{\mathscr{K}}(z)$ is generated by the curvatures alone in the generic case, if $k$ is greater than one.

3. *Equivalence of Connections.* This consists of the proof of the Equivalence Theorem, in a slightly sharpened form (Theorem II) which is useful for applications. We include an example of a one-parameter family of connections, each of which has the same curvature and covariant derivatives to order $n-1$, but none of which are equivalent, to show our results are sharp.

4. *Global Equivalence.* For bundles over $\Omega$, a manifold, we show that the local equivalences arising from the Equivalence Theorem can be extended to a global equivalence on $\Omega$ minus the coalescing set, if this complement is simply connected. We conjecture that the connectivity of this complement is the only obstruction to extending to a global equivalence on all of $\Omega$, and give some preliminary results in this direction. For example, we show that if $\Omega$ is $\mathbb{C}$ and the coalescing set is contained in a slit then local equivalence on $\mathbb{C}$ minus the coalescing set implies global equivalence on all of $\mathbb{C}$.

*Appendix.* A *generic connection* as used in the statement of the Equivalence Theorem is a connection whose curvatures generate the full $n \times n$ matrix algebra at each point (if the real dimension of the base space $\Omega$ is bigger than 2; when it equals 2 we require that the curvature have distinct eigenvalues). We show using transversality that a generic connection in this sense is generic, that is, there is an open dense subset of such connections in the space of all connections on a fixed Hermitian bundle. We also show that for a holomorphic bundle, there is an open dense subset of Hermitian structures for which the corresponding canonical connections are generic.

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**0.4.** We now review quickly and fix notation for the fundamental objects under discussion.

A *vector bundle* $E$ over a manifold $M$ (paracompact, connected) is a parameterized family of vector spaces; that is, $E$ is a manifold, there is a map $\pi: E \to M$ called *projection* such that each *fibre*, $E_x = \pi^{-1}\{x\}$, is a complex vector space for $x$ in $M$. The *dimension of* $E$ is the dimension of the fibre as a vector space over $\mathbb{C}$. A *section* $s$ of $E$ on an open set $\Omega$ in $M$ is a function from $\Omega$ into $E$ such that $s(x)$ is in $E_x$ for each $x$ in $\Omega$. A *frame* $s = (s_1, \dots, s_n)$ for $E$ on $\Omega$ is a collection of sections such that at each point $x \in \Omega$, the vectors $s_1(x), \dots, s_n(x)$ form a basis for $E_x$. As part of the definition of a vector bundle, each point in $M$ is assumed to have a neighborhood on which there exists a frame; this is the *local triviality* of a vector bundle. The bundle is $C^\infty$, real-analytic, or holomorphic in case $E$, $\pi$, and the local trivializing frames are all $C^\infty$, real-analytic, or complex analytic. Note that if $E$ is a holomorphic bundle then it is also $C^\infty$ and real-analytic, so we can talk of a $C^\infty$ section or a real-analytic frame for $E$.

If $E$ has a frame $s$ on $\Omega$ and a frame $\tilde{s}$ on $\tilde{\Omega}$, then on the overlap $\Omega \cap \tilde{\Omega}$, there is an $n \times n$ invertible matrix-valued function $A(x) = (a_{ij}(x))$ such that

$$\tilde{s} = sA \tag{0.4.1}$$

that is, $\tilde{s}_j(s) = \sum a_{ij}(x)\, s_i(x)$. The matrix $A$, called the *transition function*, is required to be $C^\infty$, real-analytic, or complex-analytic as $E$ and the frames are.

If $E$ and $\tilde{E}$ are vector bundles over $M$, then a *bundle map* $\Phi: E \to \tilde{E}$ is a map which sends $E_x$ to $\tilde{E}_x$ and is linear on $E_x$, for each $x$ in $M$. A *sub-bundle* $F$ of $E$ is a bundle over $M$, where $F$ is a submanifold of $E$, and the inclusion map of $F$ into $E$ is a bundle map. If $\Phi$ is any bundle map ($C^\infty$, real or complex-analytic) with constant rank on the $E_x$'s then the kernel bundle, $\ker \Phi = \bigcup_{x \in M} \ker \Phi|_{E_x}$, is a sub-bundle of $E$.

A *connection* $D$ on $E$ is a differential operator which takes $C^\infty$ sections of $E$ into sections with 1-form coefficients such that $D$ is linear on the vector space of all sections and satisfies the Leibnitz rule on each open set $\Omega$:

$$D(fs) = (df)\, s + fDs \tag{0.4.2}$$

for all sections $s$ and all functions $f$. If $s = (s_1, \dots, s_n)$ is a frame on $\Omega$, then

$$Ds = s\Theta \tag{0.4.3}$$

where $\Theta = (\theta_{ij})$ is an $n \times n$ matrix of $C^\infty$ 1-forms, called the *matrix of connection* 1-*forms* relative to $s$. Consistent with our notation, (0.4.3) means that $Ds_j = \sum_{i=1}^{n} \theta_{ij} s_i$. If $\tilde{s}$ is another frame, on $\tilde{\Omega}$, and $D\tilde{s} = \tilde{s}\tilde{\Theta}$ where $\tilde{s} = sA$, then

$$A\tilde{\Theta} = \Theta A + dA \tag{0.4.4}$$

by (0.4.2) and (0.4.3).

We say that $E$ is a *Hermitian vector bundle* if there is a Hermitian inner product $(\ ,\ )$ on each fibre $E_x$ which varies appropriately with $x$, that is, if $s$ is a $C^\infty$ frame and $H(x) = ((s_j(x), s_i(x)))_{i,j=1}^{n}$ is the $n \times n$ *Gramian* matrix, then $H(x)$ is to be positive definite and $C^\infty$ (or real-analytic if so specified and $s$ is real-analytic). If $\tilde{s}$ is another frame and $\tilde{H}$ is the Gramian, then $\tilde{s} = sA$ implies that

$$\tilde{H}(x) = A^*(x)\, H(x)\, A(x). \tag{0.4.5}$$

Note that even if the frame is complex-analytic, (0.4.5) precludes that the Gramian could be complex-analytic for all holomorphic frames; note also that we use "holomorphic" and "complex-analytic" interchangeably. By Gram–Schmidt, on a Hermitian bundle there always exist *orthonormal frames* $((s_i(x), s_j(x)) = \delta_{ij})$ in a neighborhood of any point.

We say that a connection $D$ on a Hermitian bundle $E$ is *metric-preserving* if the inner product satisfies

$$(Ds, t) + (s, Dt) = d(s, t) \tag{0.4.6}$$

for any two $C^\infty$ sections $s$ and $t$. It is easy to check that this is equivalent to

$$\Theta \text{ is skew-hermitian} \tag{0.4.7}$$

relative to any choice of orthonormal frame $s$.

Every bundle has a connection, this follows from a partition of unity argument. If $E$ is both holomorphic and Hermitian, then there is a natural connection $D$ on $E$, called the *canonical connection*, such that $D$ is both metric-preserving and as compatible with the complex structure as it can be, namely, if $s$ is a holomorphic section then $Ds$ is a $C^\infty$ section with 1-form coefficients of type $(1, 0)$ (i.e., if $z_1, \ldots, z_m$ are complex coordinates on a neighborhood in $M$, then $Ds = \sum (dz_i)\, s^i$ where $s^1, \ldots, s^n$ are $C^\infty$ sections—there are no $d\bar{z}_j$ terms). If $s$ is a holomorphic frame and $H$ is its Gramian, then the matrix of connection 1-forms $\Theta$ for the canonical connection relative to $s$ is determined by

$$\Theta = H^{-1}\partial H \tag{0.4.8}$$

where $\partial f \equiv \sum (\partial f/\partial z_i)\, dz_i$ and $\bar{\partial} f \equiv \sum (\partial f/\partial \bar{z}_j)\, dz_j$. Indeed the metric-preserving condition (0.4.6) yields that $dH$ equals $\Theta^* H + H\Theta$, so $\partial H$ equals

$H\Theta$ since $\Theta$ has no $d\bar{z}_j$ terms. Note that $\Theta$ itself is merely $C^\infty$, not holomorphic. This is the reason that although we are primarily interested in Hermitian holomorphic bundles, we deal with the more general case of Hermitian bundles with metric-preserving connections, since the canonical connection does not generally exhibit any holomorphic properties.

If $\mathscr{H}$ is a separable Hilbert space and $f_1,\ldots,f_n$ are holomorphic $\mathscr{H}$-valued functions on $\Omega$ a domain in $\mathbb{C}^k$, such that for each $z$ in $\Omega$ the vectors $f_1(z),\ldots,f_n(z)$ are independent in $\mathscr{H}$, then we can form a holomorphic bundle $E_f$ over $\Omega$ such that the fibre $(E_f)_z$ is just the vector space spanned by $f_1(z),\ldots,f_n(z)$. The bundle $E_f$ inherits the obvious Hermitian structure from $\mathscr{H}$. The *Grassmannian* $\mathrm{Gr}(n,\mathscr{H})$ is the collection of all $n$-dimensional subspaces of $\mathscr{H}$. A *holomorphic map* $f$ from $\Omega$ into $\mathrm{Gr}(n,\mathscr{H})$ is a function from $\Omega$ into $\mathrm{Gr}(n,\mathscr{H})$ which looks locally like the span of a collection $f_1,\ldots,f_n$ of holomorphic functions as above. Any such holomorphic map induces, in an obvious generalization of the construction above, a Hermitian holomorphic bundle $E_f$. Of particular interest to us is the case where $f$ is induced by an operator on $\mathscr{H}$, e.g., where $T$ is in the class $\mathscr{B}_n(\Omega)$ [CD1] and $f(z)$ is $\ker(T-z)$, so the fibre $(E_f)_z$ is also $\ker(T-z)$.

If $E$ and $\tilde{E}$ are Hermitian bundles over a manifold $M$, a $C^\infty$ bundle map $\Phi$ from $E$ to $\tilde{E}$ is an *isometry* if it is an isometry of $E_x$ onto $\tilde{E}_x$ for each $x$ in $M$. If $E$ and $\tilde{E}$ have connections $D$ and $\tilde{D}$, $\Phi$ is *connection-preserving* if

$$\tilde{D}(\Phi(s)) = \Phi(Ds) \tag{0.4.9}$$

for every $C^\infty$ section $s$ of $E$. Note that $Ds$ is a section with 1-form coefficients; $\Phi$ acts on the section part and is linear over the 1-forms. We say $\Phi$ is an *equivalence* of Hermitian bundles $E$ and $\tilde{E}$ with connections $D$ and $\tilde{D}$ if

$$\Phi \text{ is an isometry and connection-preserving.} \tag{0.4.10}$$

If $\Phi$ is an equivalence, then $E$ and $\tilde{E}$ have the same dimension, $\Phi$ is invertible, and $\Phi^{-1}\colon \tilde{E} \to E$ is an equivalence.

Note that if $\Phi$ is a bundle map, $s$ a frame for $E$ over $\Omega$, $\tilde{s}$ a frame for $\tilde{E}$ over $\Omega$, then $\Phi(s) = \tilde{s}A$, where $A$ is the matrix of $\Phi$. If $\Theta$ and $\tilde{\Theta}$ are the corresponding matrices of connection 1-forms then $\Phi$ is connection-preserving if and only if

$$A\Theta = \tilde{\Theta}A + dA. \tag{0.4.11}$$

From this we obtain the following trivial proposition which is the motivation for our interest in equivalence of connections.

PROPOSITION 0.5. *Let $\Phi$ be a $C^\infty$ bundle map from $E$ into $\tilde{E}$, where $E$ and $\tilde{E}$ are $n$-dimensional Hermitian holomorphic vector bundles over $M$ a* *complex manifold. Then $\Phi$ is an equivalence of Hermitian holomorphic vector bundles (i.e., a holomorphic isometry) if and only if $\Phi$ is an equivalence between the canonical connections of $E$ and $\tilde{E}$.*

*Proof.* Let $s$ and $\tilde{s}$ be holomorphic frames for $E$ and for $\tilde{E}$. If $\Phi$ preserves the connections, then by (0.4.11) $\bar{\partial}A$ is 0, so $A$ is holomorphic and hence $\Phi$ is also. Conversely, if $\Phi$ is a holomorphic isometry, let $\tilde{s}$ equal $\Phi(s)$, then $\tilde{s}$ is a holomorphic frame and $A$ is the identity matrix. Then (0.4.8) implies that $\Theta$ equals $\tilde{\Theta}$ so $\Phi$ is an equivalence of connections.

To put this result in context we recall the Rigidity Theorem [CD1, Theorem 2.2]:

PROPOSITION 0.6. *Let $\Omega$ be an open connected subset of $\mathbb{C}^k$ and $f$ and $\tilde{f}$ holomorphic maps from $\Omega$ to $\mathrm{Gr}(n, \mathscr{H})$, such that the closure of the spans $\bigvee_{\omega\in\Omega} f(\omega)$ and $\bigvee_{\omega\in\Omega} \tilde{f}(\omega)$ both equal $\mathscr{H}$. Then $f$ and $\tilde{f}$ are congruent (there is a unitary $U$ on $\mathscr{H}$ such that $\tilde{f} = U \circ f$) if and only if the corresponding bundles $E_f$ and $E_{\tilde{f}}$ are locally equivalent as Hermitian holomorphic bundles over $\Omega$.*

Combining the Equivalence Theorem and Propositions 0.5 and 0.6 leads us to a necessary and sufficient condition for two holomorphic maps into Grassmannians to be congruent in terms of a computable pointwise condition on their curvatures. This in turn gives conditions for two operators in $\mathscr{B}_n(\Omega)$ to be unitarily equivalent.

**0.7.** The fundamental invariants of a connection are obtained from its curvature. If $D$ is any connection on a $C^\infty$ vector bundle $E$ over $M$ a manifold, the curvature $K$ is the bundle map from $E$ into $E$ with 2-form coefficients given as follows:

If $s$ is a frame for $E$ over $\Omega$ open in $M$, $\Theta$ the matrix of connection 1-forms for $D$ relative to $s$, then $K$ has matrix $K(s)$ (relative to $s$) where

$$K(s) = d\Theta + \Theta \wedge \Theta. \tag{0.7.1}$$

If $\tilde{s}$ is another frame for $E$ over $\tilde{\Omega}$, and $\tilde{s} = sA$ on $\Omega \cap \tilde{\Omega}$, then (0.4.4) implies

$$AK(\tilde{s}) = K(s) A \tag{0.7.2}$$

so there actually is a bundle map $K$ represented by $K(s)$ relative to a particular choice of frame. (Technically, $K$ is a bundle map from $E$ into $E \otimes \bigwedge^2 T^*(M)$.)

Note that if $D$ is the canonical connection on a Hermitian holomorphic vector bundle $E$, and $s$ is a holomorphic frame, then (0.4.8) and (0.7.1) imply that

$$K(s) = \bar{\partial}(H^{-1}\partial H), \tag{0.7.3}$$

where $H$ is the Gramian, so $K$ is of type (1, 1), that is, $K(s)$ is locally the sum of $C^{\infty}$ matrices times the 2-forms $dz_i\, d\bar{z}_j$.

**0.8.** In general, if $x_1,\dots, x_k$ are local coordinates on the manifold $M$ then

$$K(s) = \sum_{i<j} R_{ij}\, dx_i\, dx_j \tag{0.8.1}$$

where the $R_{ij}$ are $C^{\infty}$ $n \times n$ matrix-valued functions. We are interested in determining the connection $D$ in terms of the curvatures $R_{ij}$ (actually we will use complex notation, see Section 2). In order to do this we need to discuss the algebra of all bundle maps which commute with $K$, or dually, the algebra generated by the $R_{ij}$ (and later their covariant derivatives). This we do in Section 1.

It turns out that the $R_{ij}$'s do not in general determine the connection $D$. We need their covariant derivatives as well. If $\Phi$ is a bundle map of $E$ into itself, we define $\Phi_{x_i}$, the *covariant derivative of* $\Phi$ *with respect to* $x_i$, *by*

$$[D, \Phi] = D\Phi - \Phi D = \sum_{i} \Phi_{x_i}\, dx_i \tag{0.8.2}$$

where the $\Phi_{x_i}$ are bundle maps (defined in a coordinate neighborhood in $M$) from $E$ into $E$. This procedure can be iterated to give higher-order covariant derivatives of $\Phi$. In particular we can apply this to the $R_{ij}$ to get the *covariant derivatives of the curvatures.* We need to consider the algebras generated by the $R_{ij}$, then their first-order covariant derivatives, then their second-order and so on. Again, the mechanism for doing this will be developed in Section 1 and then applied in Section 2 (where we use complex notation— all the results go through in case $M$ is a real manifold, as long as $E$ is a complex bundle).

## 1. $C^{\infty}$ Block Diagonalization

**1.1.** Let $A(x)$ be a $C^{\infty}$ $n \times n$ self-adjoint matrix-valued function defined on $\mathbb{R}^k$. If $\Omega$ is an open subset of $\mathbb{R}^k$ on which the number of distinct eigenvalues of $A(x)$ is constant, and $x_0$ is in $\Omega$, then in a neighborhood of $x_0$ the matrix $A(x)$ has a $C^{\infty}$ *diagonalization.* That is, there exists a $C^{\infty}$ $n \times n$

unitary matrix $U(x)$ such that $U(x)^{-1}A(x)\,U(x)$ is diagonal for all $x$ in a neighborhood of $x_0$:

$$U(x)^{-1}A(x)\,U(x)=\begin{bmatrix}\lambda_1(x) & & & & & 0\\ & \ddots & & & & \\ & & \lambda_1(x) & & & \\ & & & \ddots & & \\ & & & & \lambda_r(x) & \\ & & & & & \ddots \\ 0 & & & & & \lambda_r(x)\end{bmatrix} \tag{1.1.1}$$

where $\lambda_1(x) < \cdots < \lambda_r(x)$ are $C^\infty$ functions which are the distinct eigenvalues of $A(x)$; $\lambda_j(x)$ appears with constant multiplicity $m_j$ in (1.1.1).

For $k$ equal to 2, this result was the initial step in our analysis of the curvature of a Hermitian holomorphic vector bundle over a domain in $\mathbb{C}$, where the curvature can be thought of as a $C^\infty$ $n \times n$ self-adjoint matrix, times the 2-form $dzd\bar{z}$ (cf. [CD1]). When the bundle is over a domain in $\mathbb{R}^k$, the curvature is the *sum* of $C^\infty$ $n \times n$ matrices times 2-forms (0.8). This leads to the consideration of a collection $\{A_1(x),\dots, A_l(x)\}$ of $C^\infty n \times n$ matrix-valued functions defined on $\mathbb{R}^k$, such that for each $i$, $A_i^*(x) = A_j(x)$ for some $j$, that is, the collection is *self-adjoint*. We would like to show that such a self-adjoint collection has a $C^\infty$ block diagonalization,

$$U(x)^{-1}A_i(x)\,U(x)=\begin{bmatrix}A_1^i(x) & & & & & 0\\ & \ddots & & & & \\ & & A_1^i(x) & & & \\ & & & \ddots & & \\ & & & & A_r^i(x) & \\ & & & & & \ddots \\ 0 & & & & & A_r^i(x)\end{bmatrix} \tag{1.1.2}$$

for all $i$, where $U(x)$ is $C^\infty$ unitary and the $A_j^i(x)$ are $C^\infty$ matrices, appearing with constant multiplicity $m_j$ in (1.1.2).

There are two difficulties associated with this notion. The first is in what sense the $A_1^i(x),\dots, A_r^i(x)$ are supposed to be different. Indeed, for a given $i$ they could be identical. Furthermore, if $A_j^i(x)$ and $A_k^i(x)$ are unitarily equivalent for all $i$, by the same unitary, then we should change $U(x)$ so that $A_j^i(x)$ equals $A_k^i(x)$.

The second difficulty is what should be the size of the blocks $A_j^i(x)$. We could trivially satisfy (1.1.2) by taking $k=1$, $m_1=1$, and $A_1^i(x)=A_i(x)$, so we should require the $A_j^i$'s to be of minimal size as matrices.

**1.2.** Before we indicate how to resolve these difficulties we introduce some useful notation for block diagonal matrices.

DEFINITION 1.2.1. Let $m$ and $n$ be positive integers. If $A$ is a complex $n \times n$ matrix, we denote by $A \otimes I_m$ the block diagonal matrix consisting of $A$ repeated $m$ times on the diagonal. The algebra of all such matrices is denoted by $M(n, \otimes m)$, and is a subalgebra of the algebra $M(mn, \mathbb{C})$ of all complex $mn \times mn$ matrices.

For example, we have

$$A \otimes I_2 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

DEFINITION 1.2.2. *Let* $\mathscr{M}=(m_1,..., m_r)$ and $\mathscr{N}=(n_1,..., n_r)$ be $r$-tuples of positive integers. If $A_i$ is an $n_i \times n_i$ complex matrix we denote by $(A_1,..., A_r) \otimes I_{\mathscr{M}}$ the matrix $(A_1 \otimes I_{m_1}) \oplus \cdots \oplus (A_r \otimes I_{m_r})$ considered as a block diagonal matrix in $M(\sum_{i=1}^r m_i n_i, \mathbb{C})$. The algebra of all such matrices is denoted $M(\mathscr{N}, \otimes \mathscr{M})$.

For example, $M((n_1, n_2), \otimes (2, 1))$ is the algebra of all matrices of the form

$$(A_1, A_2) \otimes I_{(2,1)} = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{pmatrix}$$

where $A_1$ is $n_1 \times n_1$ and $A_2$ is $n_2 \times n_2$.

At this point we can re-examine the $C^\infty$ diagonalization of $A(x)$ with which we began 1.1. Denote by $\mathscr{A}(x)$ the algebra generated by $A(x)$ and the identity matrix. Let $\mathscr{M}=(m_1,..., m_r)$, where the $m_i$ are the multiplicities of the eigenvalues of $A(x)$. Then (1.1.1) is equivalent to

$$U(x)^{-1} \mathscr{A}(x)\, U(x) = M((1,..., 1), \otimes \mathscr{M}) \tag{1.2.3}$$

*as algebras* for all $x$ in a neighborhood of $x_0$.

Now for a collection $\{A_1(x),..., A_l(x)\}$ of $C^\infty$ $n \times n$ matrix-valued functions which is self-adjoint, let $\mathscr{A}(x)$ denote the subalgebra of $M(mn, \mathbb{C})$ generated by the $A_i(x)$'s and the identity matrix at each point $x$.

DEFINITION 1.2.4. The algebra of $n \times n$ matrices $\mathscr{A}(x)$ has a $C^\infty$ *block* *diagonalization* if there exists $U(x)$ a $C^\infty$ $n \times n$ unitary matrix-valued function and $r$-tuples $\mathscr{M}$ and $\mathscr{N}$ of positive integers such that

$$U^{-1}(x)\,\mathscr{A}(x)\,U(x) = M(\mathscr{N}, \otimes \mathscr{M}) \tag{1.2.5}$$

as algebras, for all $x$.

It is a standard result in the representation theory of $C^*$-algebras that every self-adjoint matrix algebra is a direct sum of full matrix algebras with multiplicity (see [Ma], for example). That is, for fixed $x$ there is always a unitary $U(x)$ and integers given by $\mathscr{M}$ and $\mathscr{N}$ such that (1.2.5) holds. So it is not surprising that under the mildest of hypotheses—the constancy of the dimension of $\mathscr{A}(x)$—$U(x)$ can be chosen to vary in a $C^\infty$ manner:

THEOREM I ($C^\infty$ Block Diagonalization). *Let $\{A_1(x),..., A_l(x)\}$ be a self-adjoint collection of $C^\infty$ $n \times n$ matrix-valued functions on $\mathbb{R}^k$, and let $\mathscr{A}(x)$ be the algebra generated by the $A_i(x)$ and the identity. Let $\Omega$ be a connected open subset of $\mathbb{R}^k$ on which the dimension of $\mathscr{A}(x)$, as a vector space over $\mathbb{C}$, is constant for $x$ in $\Omega$. Then there exist $r$-tuples $\mathscr{M}$ and $\mathscr{N}$ of positive integers such that for each $x_0$ in $\Omega$ there is a $C^\infty$ $n \times n$ unitary matrix-valued function $U(x)$, for which $\mathscr{A}(x)$ has a $C^\infty$ block diagonalization via $U(x)$ on a neighborhood of $x_0$.*

The proof of Theorem I is a straightforward generalization of the decomposition of a self-adjoint matrix algebra as developed in [Ma]. We include a proof for several reasons. First, the ideas and notation developed in the proof will be used throughout much of the rest of this paper. This should aid the reader who is not intimately familiar with the theory of $C^*$-algebras, especially as we deal only with a very special case—namely, that of matrix algebras. Second, in [CD1] we showed our results held except on a closed nowhere dense set about which we knew very little. We now can identify the set precisely as the set on which the dimension of a certain algebra $\mathscr{A}(x)$ is not locally constant. Finally, we need a technical generalization of Theorem I to several algebras simultaneously. Having written down a proof of Theorem I, we can then easily sketch the changes necessary for the proof of the generalization, which would otherwise be very messy.

1.3. *Remarks.* The integer $r$ is the dimension of the center of $\mathscr{A}(x)$ and is thus uniquely determined. The $r$-tuples $\mathscr{M}$ and $\mathscr{N}$ are determined up to a permutation. If $(n_1, m_1),..., (n_r, m_r)$ are ordered lexicographically, then $\mathscr{M}$ and $\mathscr{N}$ are uniquely determined by $\mathscr{A}(x)$ as well. Indeed as we shall see (1.12), there are $r$ irreducible central projections $P_1,..., P_r$ in $\mathscr{A}(x)$, with the rank of $P_i$ equal to $m_i n_i$ and the dimension of $P_i \mathscr{A}(x) P_i$ equal to $(n_i)^2$. The $P_i$'s are unique up to order, so the $m_i$'s and $n_i$'s are determined by $\mathscr{A}(x)$.

In Theorem I, if $\Omega$ is not contractible it is easy to give examples where the block diagonalization cannot be effected on all of $\Omega$ (see 1.13). Of course there will always be a diagonalization locally on $\Omega$.

DEFINITION 1.3.1. The set $\mathscr{C}$, where the dimension of $\mathscr{A}(x)$ is not locally constant (as a function of $x$), is called the *coalescing set* of $\mathscr{A}(x)$.

When $\mathscr{A}(x)$ is generated by a single self-adjoint matrix function $A(x)$, then the coalescing set is where the eigenvalues of $A(x)$ coalesce.

Theorem I holds for open sets $\Omega$ in the complement of the coalescing set. The coalescing set is trivially closed. It is nowhere dense, since if $\Omega_0$ were a non-empty open subset of the coalescing set $\mathscr{C}$, then there would exist $x_0$ in $\Omega_0$ at which dim $\mathscr{A}(x)$ attains a local maximum. But dim $\mathscr{A}(x) \geqslant \dim \mathscr{A}(x_0)$ for $x$ near $x_0$, so $x_0$ is not in $\mathscr{C}$. Thus the set $\Omega$ in the Theorem could be taken to be any component of $\mathbb{R}^k - \mathscr{C}$.

**1.4.** The proof of the Theorem consists of a series of lemmas. The first lemma is used several times to transfer information about the algebra $\mathscr{A}(x_0)$ to the algebra at nearby points.

LEMMA 1.5 (Existence of Projection Functions). *Let $P_1^0,..., P_q^0$ be disjoint projections in $\mathscr{A}(x_0)$ such that $P_1^0 + \cdots + P_q^0 = 1$. Then there exists a neighborhood $\Omega_0$ of $x_0$ and $C^\infty$ matrix-valued functions $P_1(x),..., P_q(x)$ such that the $P_i(x)$ are in $\mathscr{A}(x)$ for $x$ in $\Omega_0$; $P_i(x_0) = P_i^0$; and the $P_i(x)$ are disjoint projections summing to 1 for each $x$ in $\Omega_0$. (The $P_i$ will be called projection functions through the $P_i^0$.)*

*Proof.* Let $Q(x)$ be a polynomial in $A_1(x),..., A_l(x)$ such that $Q(x_0) = \sum_{j=1}^q jP_j^0$. By replacing $Q$ with $\frac{1}{2}(Q + Q^*)$ if necessary, we may assume $Q(x)$ is self-adjoint. Since the eigenvalues of $Q(x_0)$ are $1,..., q$ the eigenvalues of $Q(x)$ are in $\mathbb{D}_1,..., \mathbb{D}_q$ where $\mathbb{D}_j$ is the open disc in $\mathbb{C}$ of radius $\frac{1}{4}$ around $j$, for $x$ close enough to $x_0$. Now we integrate the resolvent transformation of $Q$ to obtain the $P_j$'s (cf. [RN]):

$$P_j(x) = -\frac{1}{2\pi i} \int_{\partial \mathbb{D}_j} (Q(x) - zI)^{-1}\, dz. \tag{1.5.1}$$

The $P_j$'s are $C^\infty$ by differentiation under the integral. Since we can diagonalize $Q(x)$ for fixed $x$, the Cauchy Integral Formula shows that the $P_j$'s are disjoint projections summing to 1 and that $P_j(x_0) = P_j^0$. Furthermore, for fixed $x$, $(Q(x) - zI)^{-1}$ is a polynomial in $Q(x) - zI$ and hence is in $\mathscr{A}(x)$.

LEMMA 1.6 (Constancy of Dimension). *If $P_1(x),..., P_q(x)$ are $C^\infty$*

*projection functions, disjoint and summing to* 1, *with* $P_i(x)$ *in* $\mathscr{A}(x)$ *for all* $x$ *in some open connected set* $\Omega_0 \subset \Omega$, *then*

$$\dim P_i(x)\mathscr{A}(x)P_j(x) \textit{ is constant on } \Omega_0, \tag{1.6.1}$$

*and*

$$\textit{the rank of } P_i(x) \textit{ is constant on } \Omega_0. \tag{1.6.2}$$

*Proof.* Since the $P_i(x)$ are continuous, $\dim P_i(x)\mathscr{A}(x)P_j(x) \geqslant \dim P_i(x_0)\mathscr{A}(x_0)P_j(x_0)$ for all $x$ near a fixed $x_0$. But the sum over all $i, j$ of the dimensions gives $\dim \mathscr{A}(x)$ which is constant on $\Omega$, and (1.6.1) follows. Similarly, (1.6.2) holds.

LEMMA 1.7. *Let $B$ be a self-adjoint subalgebra of $M(n, \mathbb{C})$, containing the identity. Let $r$ be the maximal number such that there are disjoint projections $P_1,\ldots, P_r$ in $B$, summing to* 1. *If the $P_i$'s commute with every element of $B$, then they form a basis for $B$ over $\mathbb{C}$.*

*Proof.* If $X$ is a normal matrix in $B$, then $P_iXP_i$ is normal for each $i$. If it were not a scalar multiple of $P_i$, then $P_i$ would be the sum of disjoint projections onto the eigenspaces of $P_iXP_i$, contradicting the maximality of $r$. Thus $X$ is a linear combination of the $P_i$'s. Since any $X$ in $B$ is a sum of normal matrices in $B$, by $X = \frac{1}{2}(X + X^*) + \frac{1}{2}(X - X^*)$, the $P_i$ form a basis.

LEMMA 1.8. *Fix $x_0$ in $\Omega$ and let $r$ be the maximal number of disjoint projections $P_1^0,\ldots, P_r^0$ summing to* 1, *with each $P_i^0$ in the center of $\mathscr{A}(x_0)$. If $P_1,\ldots, P_r$, are $C^\infty$ projection-valued functions through the $P_i^0$, defined on a connected region $\Omega_0$, with $P_i(x)$ in $\mathscr{A}(x)$ for all $x$ in $\Omega_0$, then the $P_i(x)$ are in the center of $\mathscr{A}(x)$ and form a basis for the center, for all $x$ in $\Omega_0$.*

*Proof.* By the previous Lemma, applied with $B$ equal to the center of $\mathscr{A}(x_0)$, $P_1^0,\ldots, P_r^0$ are a basis for the center. By constancy of dimension (Lemma 1.6), $P_i(x)\mathscr{A}(x)P_j(x)$ is zero at $x_0$ so identically zero in $\Omega_0$, for all $i \neq j$. Thus the $P_i(x)$ are in the center of $\mathscr{A}(x)$ for all $x$ in $\Omega_0$, and by Lemma 1.7, the dimension of the center is greater than or equal to $r$ at each $x$. Since the center is the kernel of the map "$X$ goes to $[X, \mathscr{A}_1(x)] \oplus \cdots \oplus [X, \mathscr{A}_l(x)]$" for $X$ in $\mathscr{A}(x)$, then the dimension of the center is at most $r$ for all $x$ close enough to $x_0$, where $\mathscr{A}_i(x)$ equals $P_i(x)\mathscr{A}(x)P_i(x)$. Thus the dimension is locally constant on $\Omega_0$, hence identically $r$, and the Lemma follows.

**1.9.** Let $v_1^i,\ldots, v_{q_i}^i$ be an orthonormal basis for the range of $P_i^0$. Then $P_i(x)(v_1^i),\ldots, P_i(x)(v_{q_i}^i)$ form a basis for range $P_i(x)$ for $x$ close enough to $x_0$, and using Gram–Schmidt we obtain $s_1^1(x),\ldots, s_{q_1}^1(x),\ldots,\ s_1^r(x),\ldots, s_{q_r}^r(x)$, $C^\infty$

functions from a neighborhood of $x_0$ to $\mathbb{C}^n$, such that at each $x$ they form an orthonormal basis for $\mathbb{C}^n$, and $P_j(x)\, s_p^i(x) = \delta_{ij} s_p^i(x)$. Thus if $U(x)$ is the $C^\infty$ matrix with columns $s_1^1(x),..., s_{q_r}^r(x)$, then $U(x)$ is unitary and if we put

$$\tilde{\mathscr{A}}(x) = U(x)^{-1}\, \mathscr{A}(x)\, U(x) \qquad \text{and} \qquad \tilde{P}_i(x) = U(x)^{-1}\, P_i(x)\, U(x)$$

then Lemma 1.8 states that $\tilde{\mathscr{A}}(x)$ is the direct sum *as an algebra* of the $\tilde{P}_i(x)\, \tilde{\mathscr{A}}(x)\, \tilde{P}_i(x)$. But each element of $\tilde{P}_i(x)\, \tilde{\mathscr{A}}(x)\, \tilde{P}_i(x)$ is block diagonal and has non-zero entries only in the $i$-th block on the diagonal. So $\tilde{P}_i(x)\, \tilde{\mathscr{A}}(x)\, \tilde{P}_i(x)$ can be viewed as a self-adjoint algebra of $q_i \times q_i$ matrices, generated by $\tilde{P}_i(x)\, \tilde{A}_1(x)\, \tilde{P}_i(x),..., \tilde{P}_i(x)\, \tilde{A}_l(x)\, \tilde{P}_i(x)$. Furthermore, the center of $P_i(x)\, \mathscr{A}(x)\, P_i(x)$ is one dimensional (with $P_i(x)$ as the basis element), so $\tilde{P}_i(x)\, \tilde{\mathscr{A}}(x)\, \tilde{P}_i(x)$ has center consisting of the scalar multiples of the $q_i \times q_i$ identity matrix. Thus to prove Theorem I, it suffices to show that each $\tilde{P}_i(x)\, \tilde{\mathscr{A}}(x)\, \tilde{P}_i(x)$ has a $C^\infty$ block decomposition of the form $M(n_i, \otimes\, m_i)$. We now will prove that case, and to simplify the notation we assume that $\mathscr{A}(x)$ itself has a trivial center, for each $x$ in $\Omega$.

LEMMA 1.10. *Let $\mathscr{A}(x_0)$ have a trivial center. Let $q$ be the maximal number for which there are disjoint projections $P_1^0,..., P_q^0$ in $\mathscr{A}(x_0)$, summing to* 1. *Then $P_i^0\, \mathscr{A}(x_0)\, P_j^0$ is one dimensional for all $i$ and $j$, and* $\dim \mathscr{A}(x_0)$ *is* $q^2$.

*Note.* The $P_i^0$ are not unique, which is why this case is the most complicated.

*Proof.* For each $X$ in $\mathscr{A}(x_0)$, denote by $\pi_{ij}(X)$ the matrix $P_i^0 X P_j^0$. Let $B$ be the sub-algebra of $\mathscr{A}(x_0)$ consisting of all elements which commute with the $P_i^0$'s. By Lemma 1.7, the $P_i^0$ form a basis for $B$.

We define a Hermitian form $(\,,\,)_{ij}$ on $\mathscr{A}(x_0)$ by

$$(X, Y)_{ij} = \operatorname{tr}(\pi_{ij}(X)\, \pi_{ji}(Y^*))/\operatorname{tr} P_i^0. \tag{1.10.1}$$

Since $\pi_{ij}(X)\, \pi_{ji}(Y^*)$ is in $B$, for all $i, j$ and all $X$, $Y$ in $\mathscr{A}(x_0)$, then it must be a multiple of $P_i^0$ and

$$\pi_{ij}(X)\, \pi_{ji}(Y^*) = (X, Y)_{ij} P_i^0. \tag{1.10.2}$$

But

$$(X, Y)_{ij}\, \pi_{ij}(Y) = \pi_{ij}(X)(Y^*, Y^*)_{ji} \tag{1.10.3}$$

so $\pi_{ij}(\mathscr{A}(x_0))$ is at most one-dimensional. In particular,

$$(X, X)_{ij} = (X^*, X^*)_{ji}, \qquad \text{so } \operatorname{tr} P_i^0 = \operatorname{tr} P_j^0. \tag{1.10.4}$$

Furthermore, if $\pi_{ij}(X)$ and $\pi_{jk}(Y)$ are non-zero, then so is $\pi_{ij}(X)\,\pi_{jk}(Y)$, since

$$(XP_jY, XP_jY)_{ik} = (Y, Y)_{jk}(X^*, X^*)_{ji}. \qquad (1.10.5)$$

Thus $\pi_{ik}(\mathscr{A}(x_0))$ has dimension one if $\pi_{ij}(\mathscr{A}(x_0))$ and $\pi_{jk}(\mathscr{A}(x_0))$ do. We define an equivalence relation $\sim$ by: $i \sim j$ if $\pi_{ij}(\mathscr{A}(x_0))$ is one dimensional. Reflexivity holds since $P_i^0$ is in $\pi_{ii}(\mathscr{A}(x_0))$. Assume we have numbered the $P_i^0$'s so that $1,\dots, s$ are equivalent, and if $j > s$ then $j$ is not equivalent to 1. Thus $P_i^0(\mathscr{A}(x_0))\,P_j^0$ and $P_j^0\mathscr{A}(x_0)\,P_i^0$ are both zero for all $i \leqslant s < j$ so $\sum_{i=1}^{s} P_i^0$ is in the center of $\mathscr{A}(x_0)$. This is a contradiction on the triviality of the center, unless $s$ equals $q$. Hence $P_i^0\,\mathscr{A}(x_0)\,P_j^0$ is one dimensional for all $i, j$.

**1.11.** We now complete the proof of the Theorem as follows:

We've already reduced to the case where the center of $\mathscr{A}(x)$ is trivial for all $x$ in $\Omega$. Since $\dim \mathscr{A}(x)$ is constant, by the previous Lemma it is $q^2$ for some $q$. Let $P_1(x),\dots, P_q(x)$ be projection functions in $\mathscr{A}(x)$, summing to 1, through $P_1^0,\dots, P_q^0$. Then $P_i(x)\,\mathscr{A}(x)\,P_j(x)$ has constant dimension 1.

Let $X_i$ be a non-zero element of $P_i^0\,\mathscr{A}(x_0)\,P_1^0$ for $i = 1,\dots, q$; and let $Q_i(x)$ be a polynomial in $A_1(x),\dots, A_l(x)$ such that $Q_i(x_0) = X_i$. Then $P_i(x)\,Q_i(x)\,P_1(x)$ is $C^\infty$ and non-zero for $x$ near $x_0$. Put (where we use (1.10.2) at any $x$)

$$W_i(x) = ((Q_i(x), Q_i(x))_{i1})^{-1/2}\,P_i(x)\,Q_i(x)\,P_1(x). \qquad (1.11.1)$$

Then

$$W_i(x)\,W_i^*(x) = P_i(x) \qquad \text{for } x \text{ near } x_0, \text{ for } i = 1,\dots, q. \qquad (1.11.2)$$

By (1.10.4), we have

$$W_i^*(x)\,W_i(x) = P_1(x) \qquad \text{for } x \text{ near } x_0, \text{ for } i = 1,\dots, q. \qquad (1.11.3)$$

Since $W_i(x)$ and $W_j^*(x)$ are non-zero, then as in the proof of Lemma (1.10), $W_i(x)\,W_j^*(x)$ is non-zero and hence the $W_i(x)\,W_j^*(x)$ form a basis for $\mathscr{A}(x)$ for all $x$ near $x_0$.

Let $v_1(x),\dots, v_m(x)$ be $C^\infty$ orthonormal functions from a neighborhood of $x_0$ into $\mathbb{C}^n$ such that at each $x$ they form a basis for the range of $P_1(x)$ (cf. 1.9). Define $s_1^i(x),\dots, s_m^i(x)$, $C^\infty$ functions into $\mathbb{C}^n$, by $s_j^i(x) = (W_i(x))(v_j(x))$; the $s_j^i(x)$ are in the range of $P_i(x)$, are orthonormal by (1.11.3), and form a basis for the range of $P_i(x)$ (by interchanging the roles of $P_1$ and $P_i$). Furthermore,

$$W_i(x)\,W_j^*(x)(s_p^t(x)) = \begin{cases} s_p^i(x) & \text{if } j = t \\ 0 & \text{if } j \neq t. \end{cases} \qquad (1.11.4)$$

Thus if $U(x)$ is the unitary matrix with columns $s_1^1(x),\dots, s_1^q(x),\dots, s_m^1(x),\dots, s_m^q(\mathrm{x})$ then

$$U^{-1}(x)\, W_i(x)\, W_j^*(x)\, U(x) = e_{ij} \otimes I_m \tag{1.11.5}$$

where $e_{ij}$ is the $q \times q$ matrix with 1 in the $i,j$th place, and 0 everywhere else. Since the $W_i(x)\, W_j^*(x)$ form a basis for $\mathscr{A}(x)$ this proves Theorem I when the center of $\mathscr{A}(x)$ is trivial and thus in the general case as well by 1.9.

**1.12.** The size of the blocks (the $n_i$'s) and the multiplicities (the $m_i$'s) can be taken to be constant on $\Omega$, the connected set where $\dim \mathscr{A}(x)$ is constant. This is because Lemma 1.8 shows that the maximal number $r$, of disjoint projections $P_1(x),\dots, P_r(x)$ in the center of $\mathscr{A}(x)$, is constant on $\Omega$. In addition, any projection $P$ in the center of $\mathscr{A}(x)$ is a sum of $P_i(x)$'s, hence the $P_i(x)$'s are unique up to renumbering, for each $x$. Thus in 1.9 the $P_i(x)\, \mathscr{A}(x)\, P_i(x)$ are unique up to renumbering, for each $x$. But Lemma 1.10 shows that $(n_i)^2$ is the dimension of $P_i(x)\, \mathscr{A}(x)\, P_i(x)$, so the $n_i$ are determined on all of $\Omega$. Finally, $m_i n_i$ is the rank of $P_i(x)$, so $m_i$ is determined on all of $\Omega$.

**1.13.** The result of Theorem I is purely local. We now give some examples to illustrate how $\mathscr{A}(x)$ may fail to have a global diagonalization on $\Omega$.

If $\Omega$ is simply connected it is easy to find $P_1(x),\dots, P_r(x)$, $C^\infty$ central projections in $\mathscr{A}(x)$, defined for all $x$ in $\Omega$, because the $P_i$'s exist locally and are unique up to order. If $\Omega$ is not simply connected there need not exist global central projections, much less a global diagonalization. For example, let $A_1(z)$ be the $2 \times 2$ matrix defined on $\mathbb{C}$ (in polar coordinates $z = re^{i\theta}$) by $A_1(0)$ is 0 and for $z$ non-zero

$$A_1(z) = (\exp(-r^{-2})) \begin{pmatrix} 0 & e^{i\theta} \\ 1 & 0 \end{pmatrix}.$$

Let $A_2(z)$ be $A_1(z)^*$ and let $\mathscr{A}(z)$ be the algebra generated by $A_1, A_2$, and the identity. Then $\dim \mathscr{A}(z)$ is 2 for $z$ non-zero; the coalescing set is $\{0\}$. The only non-trivial projections in $\mathscr{A}(z)$ are the central projections:

$$\frac{1}{2} \begin{pmatrix} 1 & \pm e^{i\theta/2} \\ \pm e^{-i\theta/2} & 1 \end{pmatrix}$$

so there do not exist disjoint $C^\infty$ projections $P_1(z)$ and $P_2(z)$ in $\mathscr{A}(z)$ which are defined on all of $\mathbb{C} - \{0\}$.

To give an example where there is no global diagonalization but $\Omega$ is simply connected is more complicated. The following example was shown to us by Stephen Schanuel. Let $S^2$ be the unit sphere in $\mathbb{C} \times \mathbb{R}$ and $S^3$ the unit sphere in $\mathbb{C}^2$. Then there is the Hopf fibration $\pi$ from $S^3$ to $S^2$ given by $\pi(z_1, z_2) = (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2)$. There is no *section* of the Hopf fibration, that is, no map $s: S^2 \to S^3$ such that $\pi \circ s$ is the identity. If there were a section, then $\pi^*$ would be injective on cohomology, yet $H^2(S^2, \mathbb{C})$ is one dimensional so it does not inject into $H^2(S^3, \mathbb{C})$ (which is 0).

On $\mathbb{C} \times \mathbb{R}$ let $A(\omega, t)$ be the matrix

$$A(\omega, t) = \frac{1}{2}\begin{pmatrix} 1+t & \bar{\omega} \\ \omega & 1-t \end{pmatrix}.$$

Then $A$ is $C^\infty$ and self-adjoint. It has distinct eigenvalues except at 0, so the algebra $\mathscr{A}(\omega, t)$ generated by $A$ and the identity is 2-dimensional on $\mathbb{C} \times \mathbb{R} - \{0\}$. If $\mathscr{A}$ has a diagonalization then there exists $U(\omega, t)$ a $C^\infty$ $2 \times 2$ unitary on $\mathbb{C} \times \mathbb{R} - \{0\}$ such that $UAU^{-1}$ is diagonal; we may assume that on $S^2$

$$UAU^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now let $z$ equal $\omega/(1-t)$, for $t \neq 1$ and $|\omega|^2 + t^2 = 1$, so $(\omega, t)$ is in $S^2$. Then if $V(z)$ is the $2 \times 2$ matrix

$$V(z) = \begin{pmatrix} z & 1 \\ -1 & \bar{z} \end{pmatrix}$$

we have $A(\omega, t) = V(z)^{-1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) V(z)$, so $U(\omega, t)\, V(z)^{-1}$ is diagonal for $(\omega, t)$ in $S^2$, $t \neq 1$. If $U$ has entries $u_{ij}$, then $(u_{11}(\omega, t), u_{12}(\omega, t)) = \lambda(z)\ (z, 1)$ whence $|\lambda(z)|^2 = (1 + |z|^2)^{-1}$. But then $\pi(u_{11}(\omega, t), u_{12}(\omega, t))$ equals $(|z|^2 + 1)^{-1}(2z, |z|^2 - 1)$ which is just $(\omega, t)$. Thus $\pi \circ (u_{11}, u_{12})$ is the identity on $S^2 - \{(0, 1)\}$ and hence on $S^2$. Thus a diagonalization of $\mathscr{A}(\omega, t)$ on $\mathbb{C} \times \mathbb{R} - \{0\}$ would induce a section of the Hopf fibration over $S^2$, so there can be no global diagonalization.

**1.14.** For our applications to equivalence for vector bundles we need a generalization of Theorem I.

COROLLARY I. *Let $\{A_1(x), ..., A_L(x)\}$ be a self-adjoint collection of $C^\infty$ $n \times n$ matrix-valued functions on $\mathbb{R}^k$, $\mathscr{A}(x)$ the algebra generated by $A_1(x), ..., A_l(x)$ and the identity, and $\mathscr{B}(x)$ the algebra generated by $A_1(x), ..., A_L(x)$ and the identity, where $l$ is less than $L$. Let $\Omega$ be a connected open subset of $\mathbb{R}^k$ on which the dimensions of $\mathscr{A}(x)$ and $\mathscr{B}(x)$ are constant. Then locally in $\Omega$ there is a simultaneous $C^\infty$ block diagonalization of $\mathscr{A}(x)$ and $\mathscr{B}(x)$, each block for $\mathscr{A}(x)$ being contained in a block for $\mathscr{B}(x)$.*

Note that a block for $\mathscr{A}(x)$ which has multiplicity bigger than one could be repeated in several different blocks for $\mathscr{B}(x)$ and hence the diagonalization for $\mathscr{A}(x)$ would not quite have the form (1.2.5) but would need to allow some permutations of the diagonal blocks. For example, we could have $U^{-1}(x)\mathscr{A}(x)U(x)$ equal the set of $4 \times 4$ diagonal matrices whose first and third, and second and fourth, diagonal entries are equal.

We sketch the proof: first, by Theorem I find $U(x)$ which gives a $C^\infty$ block diagonalization for $\mathscr{B}(x)$. Since $\mathscr{A}(x)$ is contained in $\mathscr{B}(x)$, any multiplicity obtained for $\mathscr{B}$ will automatically apply to $\mathscr{A}$, so we may assume that $\mathscr{B}$ has no multiplicity, that is, $U^{-1}(x)\mathscr{B}(x)U(x)$ is a direct sum of full matrix algebras $M(n_1, \mathbb{C}) \oplus \cdots \oplus M(n_r, \mathbb{C})$. Replacing $\mathscr{A}(x)$ by $U^{-1}(x)\mathscr{A}(x)U(x)$, we see that it is enough to find a $C^\infty$ diagonalization of $\mathscr{A}(x)$, where $\mathscr{A}(x)$ is a sub-algebra of $M(n_1, \mathbb{C}) \oplus \cdots \oplus M(n_r, \mathbb{C})$ for each $x$.

Let $P_1(x),..., P_s(x)$ be the maximal number of projection functions in the center of $\mathscr{A}(x)$, as in Lemma 1.8. Then each $P_i(x)$ is equal to $P_1^1 \oplus \cdots \oplus P_i^s$, where $P_i^j(x)$ is in $M(n_j, \mathbb{C})$. For fixed $j$, $P_1^j(x),..., P_s^j(x)$ are disjoint projections summing to the $n_j \times n_j$ identity, with some of the $P_i^j$'s possibly zero. Now, as in 1.9, find functions $v_1(x),..., v_l(x)$ which are orthonormal and form a basis at each $x$ for the range of $P_1^1 \oplus \cdots \oplus 0,..., P_s^1 \oplus \cdots \oplus 0,..., 0 \oplus \cdots \oplus P_1^r,..., 0 \oplus \cdots \oplus P_s^r$. Then the matrix $U(x)$ constructed in 1.9 will be in $M(n_1, \mathbb{C}) \oplus \cdots \oplus M(n_r, \mathbb{C})$. Thus $U^{-1}(x)\mathscr{B}(x)U(x)$ is still the sum of the $M(n_i, \mathbb{C})$'s and $U^{-1}(x)\mathscr{A}(x)U(x)$ is the direct sum of the $\tilde{P}_i(x)\tilde{\mathscr{A}}(x)\tilde{P}_i(x)$ as in 1.9, but here each $\tilde{P}_i(x)\tilde{\mathscr{A}}(x)\tilde{P}_i(x)$ consists of a $q_i^j \times q_i^j$ block on the diagonal in $M(n_j, \mathbb{C})$. Thus it suffices to consider only the case where $\mathscr{A}(x)$ has trivial center and is contained in $M(q_1, \mathbb{C}) \oplus \cdots \oplus M(q_r, \mathbb{C})$.

In that case, let $q^2$ be $\dim \mathscr{A}(x)$ and let $P_1(x),..., P_q(x)$ be disjoint projection functions in $\mathscr{A}(x)$ summing to 1. Then $P_i(x) = P_i^1(x) \oplus \cdots \oplus P_i^r(x)$ where $P_i^j(x)$ is in $M(q_j, \mathbb{C})$ and similarly the $W_i$'s of 1.11 satisfy $W_i = W_i^1 \oplus \cdots \oplus W_i^r$. Now let $v_1(x),..., v_m(x)$ be $C^\infty$ orthonormal functions forming a basis for the range of $P_1(x)$, grouped so that they form a basis for the range of $P_1^1(x) \oplus \cdots \oplus 0,..., 0 \oplus \cdots \oplus P_1^r(x)$. Define the $s_j^i(x)$, equal to $W_i(v_j(x))$, and $U(x)$ as in 1.11. Then $U(x)$ is in $M(q_1, \mathbb{C}) \oplus \cdots \oplus M(q_r, \mathbb{C})$, and $U^{-1}(x)\mathscr{A}(x)U(x)$ is $M(q, \otimes m)$, where $q$ divides $q_1,..., q_r$; indeed $q_i/q$ is just the dimension of the range of $P_1^i(x)$. Putting together all the reductions, this gives the joint diagonalization result.

**1.15.** We conclude this section with a result on simultaneous unitary equivalence, which will follow from Theorem I.

DEFINITION 1.15.1. Let $\{A_1,..., A_l\}$ and $\{B_1,..., B_l\}$ be self-adjoint collections of $n \times n$ matrices. They are *equivalent* if there is a *-isomorphism

$\psi$ from the algebra $\mathscr{A}$, generated by the $A_i$'s and the identity, onto the algebra $\mathscr{B}$ generated by the $B_i$'s and the identity, which satisfies:

$$\psi(A_i) = B_i \qquad \text{for all } i, \tag{1.15.2}$$

and

$$\operatorname{rank} \psi(P) = \operatorname{rank} P \qquad \text{for all projections } P \text{ in the center of } \mathscr{A}. \tag{1.15.3}$$

We show below in Proposition 1.16 that any such equivalence is *spatial*, that is, induced by unitary equivalence. Specifically, the two collections are equivalent if and only if there is an $n \times n$ unitary matrix $U$ such that $U^{-1}A_i U = B_i$ for all $i$. Note, however, that the unitary $U$ is by no means uniquely determined by the *-isomorphism $\psi$. This is why we emphasize the latter.

PROPOSITION 1.16. *Let $\mathscr{A}(x)$ and $\mathscr{B}(x)$ be the algebra of $n \times n$ matrices generated by the self-adjoint collections $\{A_1(x),..., A_l(x)\}$ and $\{B_1(x),..., B_l(x)\}$ of $n \times n$ $C^\infty$ matrix-valued functions on $\mathbb{R}^k$. Let $\Omega$ be a connected open subset of $\mathbb{R}^k$ on which the dimension of $\mathscr{A}(x)$ is constant. Then if for each $x$ in $\Omega$ the collections $\{A_1(x),..., A_l(x)\}$ and $\{B_1(x),..., B_l(x)\}$ are equivalent via $\psi(x)$, then $\psi(x)$ is $C^\infty$ and given $x_0$ in $\Omega$ there exists $U(x)$ a $C^\infty$ $n \times n$ unitary matrix in a neighborhood of $x$ so that*

$$U^{-1}(x)\, A_i(x)\, U(x) = B_i(x) \quad \textit{for all } x \textit{ near } x_0, \textit{ and for all } i. \tag{1.16.1}$$

*Note.* The special case $A_i(x)$ constant for all $x$, and the $B_i$'s the images of the $A_i$'s under some equivalence $\psi$ show that any equivalence is spatial.

*Proof.* Let $Q_1(x),..., Q_p(x)$ be polynomials in $A_1(x),..., A_l(x)$ such that $Q_1(x_0),..., Q_p(x_0)$ is a basis for $\mathscr{A}(x_0)$; then the $Q_i$'s form a basis for $\mathscr{A}(x)$ at each $x$ near $x_0$. Let $\tilde{Q}_i(x)$ be the polynomial in the $B_j$'s obtained by replacing the $A_j$'s by $B_j$'s. Then $\psi(x)(Q_i(x)) = \tilde{Q}_i(x)$ for each $x_j$; since the $\tilde{Q}_i(x)$'s are $C^\infty$, then $\psi(x)$ is $C^\infty$. That is, if $\mathscr{X}(x)$ is $C^\infty$, $\mathscr{X}(x)$ in $\mathscr{A}(x)$ for each $x$, then $\mathscr{X}(x) = \sum \alpha_i(x)\, Q_i(x)$ where the $\alpha_i$'s are $C^\infty$; so $\psi(x)(\mathscr{X}(x)) = \sum \alpha_i(x)\, \tilde{Q}_i(x)$ is $C^\infty$.

Thus if $P(x)$ is a $C^\infty$ projection-valued function, $P(x)$ in $\mathscr{A}(x)$ for each $x$ near $x_0$, then $\psi(x)(P(x))$ is a $C^\infty$ projection-valued function in $\mathscr{B}(x)$. Furthermore, $P(x)$ is in the center of $\mathscr{A}(x)$ if and only if $\psi(x)(P(x))$ is in the center of $\mathscr{B}(x)$. They have the same rank by assumption on $\psi$.

Thus in Theorem I, $r$ is the same for $\mathscr{A}(x)$ and $\mathscr{B}(x)$, so if $P_1(x),..., P_r(x)$ are the $C^\infty$ disjoint central projections for $\mathscr{A}(x)$, then $\psi(P_1(x)),..., \psi(P_r(x))$ can be taken to be the disjoint central projections for $\mathscr{B}(x)$. Since $n_i^2$ is the dimension of $P_i(x)\,\mathscr{A}(x)\,P_i(x)$, it is also the dimension of the image under $\psi$, so the $n_i$'s are the same for $\mathscr{A}(x)$ and $\mathscr{B}(x)$. Similarly, since $m_i n_i$ is the rank of $P_i(x)$, the $m_i$'s are the same for $\mathscr{A}(x)$ and $\mathscr{B}(x)$.

Thus in constructing $C^\infty$ block diagonalizations for $\mathscr{A}(x)$ and $\mathscr{B}(x)$ we can use the image under $\psi$ of the various projections of $\mathscr{A}(x)$ as the projections for $\mathscr{B}(x)$ and similarly for the $W_i(x)$'s. This shows that if $U_1(x)$ gives the diagonalization for $\mathscr{A}(x)$ and $U_2(x)$ gives the corresponding diagonalization for $\mathscr{B}(x)$, then by (1.11.5) we have

$$U_1^{-1}(x)\, XU_1(x) = U_2^{-1}(x)(\psi(x)(X))\, U_2(x)$$

for any $X$ in $\mathscr{A}(x)$, which proves (1.16.1).

Note that since $\psi(x)$ is necessarily $C^\infty$ we can choose $U(x)$ $C^\infty$ as well, although (1.16.1) could also be satisfied by a discontinuous $U(x)$.

## 2. Diagonalization of Connections

**2.1.** In this section we apply the results of Section 1 on $C^\infty$ block diagonalization to the diagonalization of a $C^\infty$ metric-preserving connection $D$ for a $C^\infty$ Hermitian vector bundle $E$ over an open connected subset $\Omega$ of $\mathbb{C}^k$. (The case of $\Omega$ in $\mathbb{R}^k$ is similar.) This decomposition is induced by covariant derivatives of the curvature off a closed nowhere dense subset of $\Omega$. Indeed, if $E$ is an $n$-dimensional bundle, then the covariant derivatives of order $n$ will suffice to induce the diagonalization, off a closed nowhere-dense subset.

A connection $D$ is a differential operator, while covariant derivatives of curvature are linear operators, and hence easier to handle. The fundamental question is to what extent the covariant derivatives determine the connection; this will be discussed in the next section.

**2.2.** For our applications it is more convenient to work with complex notation, so we now discuss curvature and covariant derivatives in this form (cf. 0.8). Since the curvature matrix $K(s)$ is a matrix of 2-forms (0.7.1), if $E$ is a $C^\infty$ bundle over $\Omega$ open contained in $\mathbb{C}^k$, then the curvature map $K$ can be written uniquely as

$$K = \sum_{i<j} \mathscr{K}_{ij}^{2,0}\, dz_i dz_j + \sum_{i,j} \mathscr{K}_{ij}^{1,1}\, dz_i d\bar{z}_j + \sum_{i<j} \mathscr{K}_{ij}^{0,2}\, d\bar{z}_i d\bar{z}_j \qquad (2.2.1)$$

where the $\mathscr{K}_{ij}^{p,q}$'s are $C^\infty$ bundle maps of $E$ into itself; their matrices $\mathscr{K}_{ij}^{p,q}(s)$ relative to a frame $s$ are just the $R_{ij}$'s in complex notation (cf. (0.8.1.)).

Note that if $D$ is the canonical connection on a Hermitian holomorphic vector bundle, then $K$ is of type (1, 1), that is, the $\mathscr{K}_{ij}^{2,0}$ and $\mathscr{K}_{ij}^{0,2}$ are all 0 by (0.7.3). Moreover, if $E$ is any bundle and $K$ is 1, $K$ is always of type (1, 1).

DEFINITION 2.3. Let $\Phi: E \to E$ be any $C^\infty$ bundle map. The covariant derivatives of $\Phi$, denoted $\Phi_{z_i}$ and $\Phi_{\bar{z}_j}$ (cf. (0.8.2.)), are defined by

$$[D, \Phi] = \sum_i \Phi_{z_i} dz_i + \sum_j \Phi_{\bar{z}_j} d\bar{z}_j$$

where $[D, \Phi]$ is $D\Phi - \Phi D$ and $\Phi$ operates on the 1-form-valued section $\omega s$ ($\omega$ a 1-form, $s$ a section of $E$) by $\Phi(\omega s) = \omega\Phi(s)$. Note that $[D, \Phi]$ is actually a bundle map of $E$ into $E$ tensored with the 1-forms, so $\Phi_{z_i}$ and $\Phi_{\bar{z}_j}$ are bundle maps.

In addition, if $\Psi$ is also a bundle map from $E$ into $E$, then $[D, \Phi\Psi] = [D, \Phi]\,\Psi + \Phi[D, \Psi]$, which implies the Leibnitz rule for covariant derivatives,

$$(\Phi\Psi)_{z_i} = \Phi_{z_i}\Psi + \Phi\Psi_{z_i} \tag{2.3.1}$$

and similarly for $\bar{z}_j$.

Covariant derivatives also behave properly under holomorphic change of coordinates, that is, if $w_1, \ldots, w_n$ also give coordinates, then

$$\Phi_{z_j} = \sum_i \Phi_{w_i} \frac{\partial w_i}{\partial z_j}. \tag{2.3.2}$$

Now $D$ acts on 1-form-valued sections by

$$D(\omega s) = (d\omega)\, s - \omega D s \tag{2.3.3}$$

where $\omega$ is a 1-form and $s$ a section of $E$. This is consistent with the definition of a connection. Indeed, (0.4.2) now holds with $s$ replaced by $\omega s$. If $s$ is a frame, then

$$\begin{aligned} D^2 s_j &= D\left(\sum_i \theta_{ij} s_i\right) \\ &= \sum_i d\theta_{ij} s_i - \sum_{i,k} \theta_{ij}\theta_{ki} s_k \\ &= \sum_k \left(d\theta_{kj} + \sum_i \theta_{ki}\theta_{ij}\right) s_k \end{aligned}$$

by anti-commutativity of 1-forms, so (0.7.1) implies that $D^2(s) = K(s)$ or that $D^2$ is a bundle map and

$$D^2 = K. \tag{2.3.4}$$

Thus for any $C^\infty$ bundle map $\Phi$ of $E$ into $E$,

$$\begin{aligned}[K, \Phi] &= D[D, \Phi] + [D, \Phi]\, D \\ &= \sum_i [D, \Phi_{z_i}]\, dz_i + \sum_j [D, \Phi_{\bar z_j}]\, d\bar z_j\end{aligned}$$

by (2.3.4) and Definition 2.3 which implies the following commutation relations for mixed covariant derivatives:

$$\begin{aligned}\mathscr{K}^{2,0}_{ij} &= (\Phi_{z_j})_{z_i} - (\Phi_{z_i})_{z_j} \\ \mathscr{K}^{1,1}_{ij} &= (\Phi_{\bar z_j})_{z_i} - (\Phi_{z_i})_{\bar z_j} \\ \mathscr{K}^{0,2}_{ij} &= (\Phi_{\bar z_j})_{\bar z_i} - (\Phi_{\bar z_i})_{\bar z_j}.\end{aligned} \tag{2.3.5}$$

In particular, we can define the various covariant derivatives of the curvature, for example, $((\mathscr{K}^{1,1}_{ij})_{z_k})_{\bar z_l}$ and the order in which we differentiate is not critical.

If $s$ is a frame, $\Phi(s)$ the matrix of $\Phi$ relative to the frame $s$, then the matrix of $[D, \Phi]$ relative to $s$ is

$$d\Phi(s) + [\Theta, \Phi(s)] \tag{2.3.6}$$

for $\Theta$ the matrix of connection 1-forms. Thus if $E$ has a Hermitian structure and if $D$ is metric-preserving, then

$$(\Phi_{z_i})^* = (\Phi^*)_{\bar z_i} \qquad \text{and} \qquad (\Phi_{\bar z_j})^* = (\Phi^*)_{z_j} \tag{2.3.7}$$

follows by applying (2.3.5) when $s$ is an orthonormal frame, since $\Theta$ is then skew-symmetric. Furthermore, in this case $K$ is skew-adjoint by (0.7.1) and the skew-symmetry of $\Theta$, so by (2.2.1) we have

$$(\mathscr{K}^{2,0}_{ij})^* = -\mathscr{K}^{0,2}_{ij} \qquad \text{and} \qquad (\mathscr{K}^{1,1}_{ij})^* = \mathscr{K}^{1,1}_{ji}. \tag{2.3.8}$$

**2.4.** We denote by $\Gamma^{\mathscr{K}}$ the set of all $\mathscr{K}^{p,q}_{ij}$'s and their covariant derivatives to all orders, for $E$ a $C^\infty$ Hermitian vector bundle with metric-preserving connection $D$ over an open subset $\Omega$ of $\mathbb{C}^k$; $\Gamma^{\mathscr{K}}$ is a subset of the algebra of all $C^\infty$ bundle maps of $E$ into itself. For each $z$ in $\Omega$, we denote by $\Gamma^{\mathscr{K}}(z)$ the restriction of the elements of $\Gamma^{\mathscr{K}}$ to the fibre $E_z$; $\Gamma^{\mathscr{K}}(z)$ is a subset of the algebra $\operatorname{End}(E_z)$ of all homomorphisms of $E_z$ into itself. We let $\mathscr{A}^{\mathscr{K}}(z)$ be the sub-algebra generated by $\Gamma^{\mathscr{K}}(z)$ and the identity. Note that $\mathscr{A}^{\mathscr{K}}(z)$ depends only on the connection $D$, not on the choice of coordinates in $\Omega$, by (2.3.2). Thus $\mathscr{A}^{\mathscr{K}}(z)$ can be defined when $\Omega$ is a manifold as well.

Let $\chi$ be a bundle map of $E$ into itself and $s$ a frame for $E$ over an open subset of $\Omega$. We denote by $\chi(s)$ the matrix of $\chi$ relative to the frame $s$. For an orthonormal frame $s$, we let $\mathscr{A}(\mathscr{K}, s)(z)$ denote the algebra of $n \times n$ matrices generated by the identity and by the $\chi(s)|_z$, for $\chi$ in $\Gamma^{\mathscr{K}}$. The algebra $\mathscr{A}(\mathscr{K}, s)(z)$ is a representation of $\mathscr{A}^{\mathscr{K}}(z)$ which does not depend on the choice of coordinates, and a different choice of frame leads to a unitarily equivalent representation.

DEFINITION 2.4.1. Let $D$ be a metric-preserving connection for a Hermitian vector bundle $E$ defined over an open connected subset $\Omega$ of $\mathbb{C}^k$. The *coalescing set* $\mathscr{C}$ of the curvature is the set of all $z$ in $\Omega$ for which $\dim \mathscr{A}^{\mathscr{K}}(z)$ is not locally constant.

If $s$ is an orthonormal frame for $E$ in a neighborhood of $z_0$, then the intersection of the coalescing set of the curvature with the neighborhood is just the coalescing set of the algebra $\mathscr{A}(\mathscr{K}, s)(z)$ (see Definition 1.3.1). Thus the coalescing set of the curvature is closed and nowhere dense.

Theorem I shows that there is a frame $s$ such that $\mathscr{A}(\mathscr{K}, s)(z)$ is block diagonal. We show that we can choose a frame such that the connection will also be block diagonal.

PROPOSITION 2.5 (Diagonalization of Connections). *Let $E$ be a $C^\infty$ Hermitian vector bundle of dimension $n$ over an open subset $\Omega$ in $\mathbb{C}^k$, with metric-preserving connection $D$. Let $z_0$ be in $\Omega$, $z_0$ not in the coalescing set for the curvature. Then there exists a neighborhood $\Omega_0$ of $z_0$ in $\Omega$ and a $C^\infty$ orthonormal frame $s$ for $E$ over $\Omega_0$ with the properties:*

$$\mathscr{A}(\mathscr{K}, s)(z) = M(\mathscr{N}, \otimes \mathscr{M}) \qquad \textit{for all } z \textit{ in } \Omega_0, \tag{2.5.1}$$

*and*

$$\Theta(s) = (\Theta_1, \ldots, \Theta_r) \otimes I_{\mathscr{M}} \tag{2.5.2}$$

*where $\Theta(s)$ is the matrix of connection 1-forms of $D$ relative to the frame $s$ and the $\Theta_i$ are $C^\infty n_i \times n_i$ matrices with 1-form coefficients defined on $\Omega_0$.*

*Proof.* Let $s$ be any orthonormal frame for $E$ in a neighborhood of $z_0$. Since $\mathscr{A}(\mathscr{K}, s)(z_0)$ is finite dimensional, there exist $\chi_1, \ldots, \chi_l$ in $\Gamma^{\mathscr{K}}$ such that $\{\chi_1(s), \ldots, \chi_l(s)\}$ at $z$ is a self-adjoint collection generating $\mathscr{A}(\mathscr{K}, s)$ $(z)$ for all $z$ in a neighborhood of $z_0$. Let $U(z)$ be a $C^\infty$ $n \times n$ unitary matrix-valued function which gives the block diagonalization of $\mathscr{A}(\mathscr{K}, s)(z)$. Let $\tilde{s}$ be a new frame defined by $\tilde{s} = sU$. Then $\mathscr{A}(\mathscr{K}, \tilde{s})(z)$ is equal to $U(z)^{-1}\mathscr{A}(\mathscr{K}, s)(z)\, U(z)$ which proves (2.5.1).

We now assume we have an orthonormal frame $s$ which satisfies (2.5.1) and seek to modify it to satisfy (2.5.2) as well.

*Step* 1. Let $P_i$ be the $i$th central projection in $M(\mathscr{N}, \otimes \mathscr{M})$, that is, $P_i$ has 1's on the diagonal in $M(\mathscr{N}_i, \otimes \mathscr{M}_i)$ and 0's everywhere else. By (2.5.1) there exists $\chi_i$ in $\Gamma^{\mathscr{K}}$ such that the matrix $\chi_i(s)$ is identically equal to $P_i$ as a function of $z$, for $z$ near $z_0$. Since $(\chi_i)_{z_j}$ and $(\chi_i)_{\bar{z}_j}$, the covariant derivatives of $\chi_i$, are also in $\Gamma^{\mathscr{H}}$, then the matrix $[D, \chi_i](s)$ is in $\mathscr{A}(\mathscr{H}, s)(z)$ tensored with the 1-forms, by Definition 2.3. But $[D, \chi_i](s)$ equals $d\chi_i(s) + [\Theta(s), \chi_i(s)]$ by (2.3.6), which is just $[\Theta(s), P_i]$. Let $\Theta_{ij}$ denote $P_i\Theta(s)P_j$. Then $[\Theta(s), P_i]$ is the sum over $j$ of the $\Theta_{ji} - \Theta_{ij}$ and is supposed to be in $M(\mathscr{N}, \otimes \mathscr{M})$ tensored with the 1-forms. Thus $\Theta_{ij}$ is zero for $i$ not equal to $j$ and it suffices to prove (2.5.2) when $r$ is one, that is, when $\mathscr{A}(\mathscr{H}, s)(z)$ equals $M(n, \otimes m)$ (so we have changed notation and now have the dimension of $E$ is $mn$).

*Step* 2. Let $A$ be a fixed $n \times n$ matrix. Then as in step 1, there is $\chi$ in $\Gamma^{\mathscr{H}}$ such that $\chi(s)$ is identically equal to $A \otimes I_m$. As above we obtain that $[\Theta(s), A \otimes I_m]$ is in $M(n, \otimes m)$ tensored with the 1-forms. Let $\Theta(s) = (\Theta_{ij})$ where each $\Theta_{ij}$ is an $n \times n$ matrix with 1-form coefficients and $1 \leqslant i, j \leqslant m$. Then in order to have $[\Theta(s), A \otimes I_m]$ be block diagonal with multiplicity $m$, we must have $[\Theta_{ij}, A]$ equal to zero for $i$ not equal to $j$, and the $[\Theta_{ii}, A]$ are all equal. This holds true for all $A$, that is $\Theta_{ij}$ and $\Theta_{ii} - \Theta_{jj}$ commute with all $A$. Thus there exist unique 1-forms $\mu_{ij}$ such that $\Theta_{ij} = \mu_{ij} I_n$ for $i$ not equal to $j$, and $\Theta_{ii} - \Theta_{11} = \mu_{ii} I$ for $i$ equal to $1, \ldots, m$. Let $\mu$ be the $m \times m$ matrix of 1-forms with coefficients $\mu_{ij}$. Denote by $I_n \otimes \mu$ the matrix of 1-forms whose $i,j$th $n \times n$ block is $\mu_{ij} I_n$. Let $\lambda$ equal $\Theta_{11}$, an $n \times n$ matrix of 1-forms. Then

$$\Theta(s) = \lambda \otimes I_m + I_n \otimes \mu. \tag{2.5.3}$$

Since $D$ is metric-preserving, $\Theta(s)$ is skew-adjoint, and hence so are $\lambda$ and $\mu$ as well.

From (0.7.1) we obtain that the 2-form-valued matrix $K(s)$ satisfies

$$K(s) = d\lambda \otimes I_m + I_n \otimes d\mu + (\lambda \wedge \lambda) \otimes I_m + I_n \otimes (\mu \wedge \mu)$$

since $(\lambda \otimes I_m) \wedge (I_n \otimes \mu) = -(I_n \otimes \mu) \wedge (\lambda \otimes I_m)$. But $K(s)$ is in $M(n, \otimes m)$ tensored with the 2-forms since each of the $K_{ij}^{p,q}(s)$'s is in $M(n, \otimes m)$. Thus $I_n \otimes (d\mu + \mu \wedge \mu)$ must be in $M(n, \otimes m)$ tensored with the two forms. Since the intersection of $I_n \otimes M(m, \mathbb{C})$ with $M(n, \mathbb{C}) \otimes I_m$ is just the scalars times the $nm$ identity matrix, then $d\mu + \mu \wedge \mu = \tau I_m$ where $\tau$ is a 2-form. Of course $\lambda$ and $\mu$ are not uniquely determined by (2.5.3).

Since $\operatorname{tr}(d\mu + \mu \wedge \mu)$ is just $d(\operatorname{tr} \mu)$, the $\operatorname{tr}(\mu \wedge \mu)$ term dropping out because of anti-commutativity of 1-forms, we have $\tau$ equal to $d((1/m) \operatorname{tr} \mu)$. Thus if we replace $\mu$ by $\mu - ((1/m) \operatorname{tr} \mu) I_m$ and $\lambda$ by $\lambda + ((1/m) \operatorname{tr} \mu) I_n$ we may assume we have chosen $\mu$ and $\lambda$ skew-adjoint and satisfying (2.5.3) such that $\mu$ satisfies the integrability condition

$$d\mu + \mu \wedge \mu = 0. \tag{2.5.4}$$

That is, if we think of $\mu$ as defining a connection, then its curvature is zero, i.e., $\mu$ gives a flat connection. In other words, there exists a $C^\infty$ $m \times m$ unitary matrix $U$ in a neighborhood of $z_0$ such that

$$\mu = -(dU)\, U^{-1}. \tag{2.5.5}$$

(This is standard and follows from the Frobenius theorem [F, p. 102].)

Now let $\tilde{s}$ be a new orthonormal frame for $E$ such that the change of frame from $s$ to $\tilde{s}$ is given by $I_n \otimes U$. Since $D$ acts as a derivation, we have under change of frame:

$$\begin{aligned}(I_n \otimes U)\, \Theta(\tilde{s}) &= \Theta(s)(I_n \otimes U) + d(I_n \otimes U)\\ &= (\lambda \otimes I_m + I_n \otimes \mu)(I_n \otimes U) - I_n \otimes \mu U\\ &= (\lambda \otimes I_m)(I_n \otimes U)\\ &= (I_m \otimes U)(\lambda \otimes I_m).\end{aligned}$$

Thus $\Theta(\tilde{s}) = \lambda \otimes I_m$ which proves the Proposition in the special case and by step 1 we are done.

**2.6.** The algebra $\mathscr{A}^{\mathscr{K}}(z)$ is generated by *all* the covariant derivatives of curvature. For any particular $z$ the order of covariant derivative needed to generate $\mathscr{A}^{\mathscr{K}}(z)$ could be large. Generically, however, order $n-1$ suffices, and this is sharp, as we shall see.

Consistent with the notation in 2.4, we let $\Gamma_j^{\mathscr{K}}$ denote the set of covariant derivatives of curvature of total order less than or equal to $j$; $\Gamma_j^{\mathscr{K}}(z)$, $\mathscr{A}_j^{\mathscr{K}}(z)$, and $\mathscr{A}_j(\mathscr{K}, s)(z)$ are all defined similarly (cf. 2.4).

For example, let $n$ be greater than 1. Fix a real number $\alpha$ and define a connection $D_\alpha$ on the trivial $n$-dimensional Hermitian vector bundle $E$ over $\mathbb{C}$ as follows. Fix an orthonormal frame $s$, let $z = x + iy$, and let $A_\alpha(z)$ be the $n \times n$ real symmetric matrix

$$A_\alpha(z) = \begin{bmatrix} x & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & & \vdots \\ 0 & 1 & 0 & \ddots & \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & & 1 & \alpha \end{bmatrix}. \tag{2.6.1}$$

The matrix of connection 1-forms for $D_\alpha$ is defined by

$$\Theta_\alpha(s) = A_\alpha(z)(dz - d\bar{z}) = 2iA_\alpha(z)\, dy \tag{2.6.2}$$

and can be viewed as the canonical connection on a holomorphic Hermitian bundle over $\mathbb{C}$ [CD1, 3.2.3] though not the trivial holomorphic Hermitian bundle.

Now let $\mathscr{K}_\alpha$ denote the curvature of $D_\alpha$. Then

$$K_\alpha(s) = d\Theta_\alpha(s) + \Theta_\alpha(s) \wedge \Theta_\alpha(s) = 2i \frac{\partial A_\alpha}{\partial x} dxdy,$$

$$\text{so } \mathscr{K}_\alpha(s) \text{ equals } \frac{\partial A_\alpha}{\partial x}$$

and hence

$$\mathscr{K}_\alpha(s) \equiv e_{11} \tag{2.6.3}$$

where $e_{ij}$ is the $n \times n$ matrix with a 1 in the $i,j$th coordinate, zero everywhere else. Similarly, $[D, \mathscr{K}_\alpha](s) = d\mathscr{K}_\alpha(s) + [\Theta(s), \mathscr{K}_\alpha(s)] = (e_{21} - e_{12})(dz - d\bar{z})$, so

$$(\mathscr{K}_\alpha)_z(s) = e_{21} - e_{12} \qquad \text{and} \qquad (\mathscr{K}_\alpha)_{\bar{z}}(s) = e_{12} - e_{21} \tag{2.6.4}$$

by Definition 2.3. Thus $\mathscr{A}_0(\mathscr{K}_\alpha, s)(z)$ equals $M((1,1), \otimes (1, n-1))$ and $\mathscr{A}_1(\mathscr{K}_\alpha, s)(z)$ equals $M((2,1), \otimes (1, n-2))$.

PROPOSITION 2.7. *If $p+q$ is less than $n$, then $(\mathscr{K}_\alpha)_{z^p\bar{z}^q}$ is independent of $\alpha$. The algebra $\mathscr{A}_j(\mathscr{K}_\alpha, s)(z)$ is independent of $\alpha$ for all $j$ and*

$$\mathscr{A}_j(\mathscr{K}_\alpha, s)(z) = \begin{cases} M((j+1, 1), \otimes (1, n-j-1)) \text{ for } 0 \leqslant j \leqslant n-2 \\ M(n, \mathbb{C}) \text{ for } j \geqslant n-1. \end{cases} \tag{2.7.1}$$

*Note that when $p+q$ equals $n$, $(\mathscr{K}_\alpha)_{z^p\bar{z}^q}$ itself does depend on $\alpha$. For example, when $n$ is 3, we have*

$$(\mathscr{K}_\alpha)_{z^2}(s) = \begin{pmatrix} 2 & -x & 1 \\ -x & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \textit{but} \quad (\mathscr{K}_\alpha)_{z^2\bar{z}} = \begin{pmatrix} 0 & x^2 + 4\frac{1}{2} & \alpha - 2x \\ -x^2 - 5\frac{1}{2} & 0 & -3 \\ 2x - \alpha & 3 & 0 \end{pmatrix}.$$

*Proof.* Assume true for $j-1$, where $1 \leqslant j \leqslant n-1$, and for $p+q \leqslant j-1$. Let $\chi = (\mathscr{K}_\alpha)_{z^p\bar{z}^q}$, with $p+q \leqslant j-1$. Then $\chi_z(s)$ equals $\partial\chi(s)/\partial z + [A_\alpha, \chi(s)]$. But $\partial\chi(s)/\partial z$ is independent of $\alpha$ and is in $M((j,1), \otimes (1, n-j))$ since $\chi(s)$ is. For fixed $z$, $\chi(s)$ is of the form $\left(\begin{smallmatrix} B & 0 \\ 0 & cI \end{smallmatrix}\right)$, for $B$ of size $j \times j$, $c \in \mathbb{C}$. Since $A_\alpha$ has blocks $A_{ij}$ $(i,j = 1,2)$ where $A_{11}$ is a $j \times j$ matrix, $A_{22}$ is $(n-j) \times (n-j)$ and

$$A_{11} = \begin{pmatrix} x & 1 & \cdots & 0 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \qquad A_{12} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{pmatrix}, \qquad A_{21} = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

and

$$A_{22} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & & \ddots & \\ & \ddots & & 1 \\ 0 & & 1 & \alpha \end{pmatrix},$$

then

$$[A_\alpha, \chi(s)] = \begin{pmatrix} [A_{11}, B] & cA_{12} - BA_{12} \\ A_{21}B - cA_{21} & 0 \end{pmatrix},$$

which doesn't involve $\alpha$. Furthermore, $cA_{12} - BA_{12}$ has all columns zero but the first and $A_{21}B - cA_{21}$ has all rows zero but the first. Thus $[A_\alpha, \chi(s)]$ is in $M((j+1,1), \otimes\, (1, n-j-1))$ and hence so is $\chi_z(s)$. Since $\chi_{\bar{z}} = \chi_z^*$, we have only to show that $\mathscr{A}_j(\mathscr{K}_\alpha, s)(z)$ generates all of $M((j+1,1), \otimes\,(1, n-j-1))$. So let $X$ be a polynomial in the $\mathscr{K}_{z^p\bar{z}^q}$'s with $p+q \leqslant n-1$, such that at $z$ we have

$$X(s) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 1 & & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \qquad (j\text{th row});$$

such an $X$ exists by the induction hypothesis. Then

$$X_z = \frac{\partial X(s)}{\partial z} + [A_\alpha, X(s)] = [A_\alpha, X(s)] = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \qquad ((j+1)\text{st row}).$$

Since $X(s)$ is in $\mathscr{A}_{j-1}(\mathscr{K}_\alpha, s)(z)$, then $X_z$ and $X_{\bar{z}}\,(=X_z^*)$ together with $\mathscr{A}_{j-1}(\mathscr{K}_\alpha, s)(z)$ generate $M((j+1,1), \otimes\,(1, n-j-1))$, which proves the proposition.

In this example, the $(n-1)$st-order covariant derivatives generate all of $\mathscr{A}^{\mathscr{K}}(z)$. No lower order will suffice. We show now that this is sufficient for any connection. Then we will be able to show that $n$th-order covariant derivatives of the curvature suffice to determine the connection. This is the "one more derivative" phenomenon. In our example it is the $n$th-order derivatives which carry the dependence on $\alpha$.

**2.8.** We first discuss chains of matrix algebras obtained by taking commutators with fixed matrices; this is in fact how $\mathscr{A}_{j+1}(\mathscr{H}, s)(z)$ is obtained from $\mathscr{A}_j(\mathscr{H}, s)(z)$, as we shall see.

If $\mathscr{A}$ is any self-adjoint matrix algebra containing the identity, we denote by $\#\mathscr{A}$ the total number of diagonal blocks (counting multiplicity) in the block diagonalization of $\mathscr{A}$. Thus if $\mathscr{A}$ has a block diagonalization as $M(\mathscr{N}, \otimes \mathscr{M})$ with $\mathscr{N}$ and $\mathscr{M}$ $r$-tuples of integers, then

$$\#\mathscr{A} = m_1 + \cdots + m_r. \tag{2.8.1}$$

Let $\mathscr{A}$ be a self-adjoint algebra of $n \times n$ matrices, containing the identity, and $\{X_1, ..., X_s\}$ a self-adjoint collection of $n \times n$ matrices. We set $\mathscr{A}_0$ equal to $\mathscr{A}$, and let $\mathscr{A}_j$ be the algebra generated by $\mathscr{A}_{j-1}$ and $\{[A, X_i] | A \in \mathscr{A}_{j-1}, 1 \leqslant i \leqslant s\}$ for $j$ bigger than 0. Clearly if $\mathscr{A}_j$ equals $\mathscr{A}_{j+1}$, then $\mathscr{A}_j$ equals $\mathscr{A}_k$ for all $k$ bigger than $j$. We denote by $J(\mathscr{A})$ the first $j$ for which this happens. We wish to show that $J(\mathscr{A})$ is less than $n$.

Note that the $\mathscr{A}_j$'s for $0 \leqslant j \leqslant J(\mathscr{A})$ form an increasing chain of algebras in $M(n, \mathbb{C})$ which a priori could have length $2n-1$, e.g., $M(1, \otimes n) \subset M((1, 1), \otimes (1, n-1)) \subset \cdots \subset M((1, ..., 1), \otimes (1, ..., 1)) \subset M((2, 1, ... 1), \otimes (1, ..., 1)) \subset \cdots \subset M((n-1, 1), \otimes (1, 1)) \subset M(n, \mathbb{C})$. The problem is that although

$$\#\mathscr{A}_j \geqslant \#\mathscr{A}_{j+1} \tag{2.8.2}$$

equality can hold without forcing $j$ to be $J(\mathscr{A})$. However, if $\#\mathscr{A}_j = \#\mathscr{A}_{j+1}$ then we get some information about the $X_i$'s.

LEMMA 2.9. *If $\#\mathscr{A}_j$ equals $\#\mathscr{A}_{j+1}$, then each $X_i$ decomposes into*

$$X_i = Y_i + Z_i \tag{2.9.1}$$

*where $Y_i$ is in $\mathscr{A}_{j+1}$ and $Z_i$ commutes with $\mathscr{A}_j$.*

*Proof.* Since we can simultaneously block diagonalize $\mathscr{A}_j$ and $\mathscr{A}_{j+1}$, it suffices to assume that $\mathscr{A}_j$ equals $M(N, \otimes \mathscr{M})$ for some $r$-tuples $\mathscr{M}$ and $\mathscr{N}$ and that $\mathscr{A}_{j+1}$ is block diagonal with the same size blocks as $\mathscr{A}_j$, but possibly different multiplicities. The rest of the argument is along the same lines as the proof of Proposition 2.5 and is omitted. (Note that (2.9.1) generalizes (2.5.3).)

LEMMA 2.10. *Let $\mathscr{A}$ be an algebra of $n \times n$ matrices.* (i) *If $\#\mathscr{A}_0$ and $\#\mathscr{A}_1$ both equal n, then $\mathscr{A}_1$ equals $\mathscr{A}_0$ so $J(\mathscr{A})$ is* 0. (ii) *If $\#\mathscr{A}_{j-1}$, $\#\mathscr{A}_j$, and $\#\mathscr{A}_{j+1}$ are all equal, then $\mathscr{A}_{j+1}$ equals $\mathscr{A}_j$, so j is greater than or equal to $J(\mathscr{A})$.* (iii) *If $0 < j < J(\mathscr{A})$ and $\#\mathscr{A}_j$ equals $\#\mathscr{A}_{j+1}$, then $\#\mathscr{A}_{j-1}$ is at least two more than $\#\mathscr{A}_{j+1}$.*

*Proof.* (i) If $\#\mathscr{A}_0$ and $\#\mathscr{A}_1$ equal $n$, then $\mathscr{A}_0$ and $\mathscr{A}_1$ are both subalgebras of the diagonal $n \times n$ matrices. Thus (i) is trivial (by Lemma 2.9).

(ii) By Lemma 2.9, $X_i$ can be decomposed into $Y_i + Z_i$ where $Y_i$ is in $\mathscr{A}_{j+1}$ and $Z_i$ commutes with $\mathscr{A}_j$. It can also be decomposed into $\tilde{Y}_i$ and $\tilde{Z}_i$, where $\tilde{Y}_i$ is in $\mathscr{A}_j$ and $\tilde{Z}_i$ commutes with $\mathscr{A}_{j-1}$. Let $W_i$ equal $Y_i - \tilde{Y}_i$. Thus $W_i$ is in $\mathscr{A}_{j+1}$. But $W_i$ equals $\tilde{Z}_i - Z_i$, so it commutes with $\mathscr{A}_{j-1}$. Any block diagonal matrix which commutes with an algebra of block diagonal matrices with the same size blocks must have diagonal blocks which are each scalar multiples of the identity. Thus by simultaneously diagonalizing $\mathscr{A}_{j-1}$, $\mathscr{A}_j$, and $\mathscr{A}_{j+1}$, we see that $W_i$ commutes with $\mathscr{A}_j$ as well. Thus for any $A$ in $\mathscr{A}_j$, $[A, X_i]$, $[A, Y_i]$, and $[A, \tilde{Y}_i]$ are all equal, so $\mathscr{A}_{j+1}$ is contained in $\mathscr{A}_j$. (iii) By (ii), $\#\mathscr{A}_{j-1}$ must be greater than $\#\mathscr{A}_j$. So assume $\#\mathscr{A}_{j-1}$ is precisely $1 + \#\mathscr{A}_j$. We simultaneously block diagonalize $\mathscr{A}_{j-1}, \mathscr{A}_j$, and $\mathscr{A}_{j+1}$. Since each block of $\mathscr{A}_{j-1}$ is contained in a block of $\mathscr{A}_j$, we must have one block of $\mathscr{A}_j$ which contains exactly two blocks of $\mathscr{A}_{j-1}$ and all other blocks of $\mathscr{A}_{j-1}$ are the same size as the blocks of $\mathscr{A}_j$. By assumption, this one block in $\mathscr{A}_j$ can occur only with multiplicity one.

Let $P$ be the projection in the center of $\mathscr{A}_j$ which corresponds to this block; that is, such that $P\mathscr{A}_jP$ is the block. Let $Q$ be $1 - P$. Since $\#\mathscr{A}_j$ equals $\#\mathscr{A}_{j+1}$, $P$ is also in the center of $\mathscr{A}_{j+1}$. Thus $P$ and $Q$ commute with the $[P, X_i]$'s so $X_i$ equals $PX_iP + QX_iQ$.

Thus if we let $\mathscr{B}_0$ be the algebra of matrices $Q\mathscr{A}_{j-1}Q$, then $Q\mathscr{A}_jQ$ and $Q\mathscr{A}_{j+1}Q$ are just $\mathscr{B}_1$ and $\mathscr{B}_2$ (where we take commutators with the $QX_iQ$). But $\#\mathscr{B}_0$, $\#\mathscr{B}_1, \#\mathscr{B}_2$ are all equal, so $\mathscr{B}_2$ equals $\mathscr{B}_1$. This implies that $[QAQ, QX_iQ]$ is in $Q\mathscr{A}_jQ$ for all $A$ in $\mathscr{A}_j$. Since $[PAP, PX_iP]$ is in $P\mathscr{A}_{j+1}P$ which equals $P\mathscr{A}_jP$, we have shown that $[A, X_i]$ is in $\mathscr{A}_j$ for all $A$ in $\mathscr{A}_j$. Thus $\mathscr{A}_{j+1}$ equals $\mathscr{A}_j$, contrary to the assumption that $j$ is less than $J(\mathscr{A})$.

We can now sum up this section in the following Proposition.

PROPOSITION 2.11. *Let $\mathscr{A}$ be a self-adjoint algebra of $n \times n$ matrices containing the identity and let $\{X_1, ..., X_k\}$ be a self-adjoint collection of $n \times n$ matrices. If $\mathscr{A}_0$ equals $\mathscr{A}$, and for $j \geqslant 1$ $\mathscr{A}_j$ is the algebra generated by $\mathscr{A}_{j-1}$ and the commutators of the elements in $\mathscr{A}_{j-1}$ and the $X_i$'s, then $\mathscr{A}_{j+1}$ equals $\mathscr{A}_j$ for some j less than n. If the first such j is denoted $J(\mathscr{A})$ then $0 \leqslant J(\mathscr{A}) \leqslant n-1$.*

*Proof.* It is an easy induction, using (2.8.2) and Lemma 2.10 to show *a fortiori* that

$$\#\mathscr{A}_j \leqslant n-j \qquad \text{for} \quad 0 \leqslant j \leqslant J(\mathscr{A}). \tag{2.11.1}$$

But then $\#\mathscr{A}_j$ is at least 1, so $j$ is at most $n-1$.

COROLLARY 2.12. *If $J(\mathscr{A})$ equals $n-1$, then $\mathscr{A}_{n-2}$ has no multiplicity in its block diagonalization, for $n \geqslant 2$.*

*Proof.* If $\mathscr{A}_{n-2}$ has multiplicity, then by (2.11.1), $\#\mathscr{A}_{n-2}$ is 2, so $\mathscr{A}_{n-2}$ has only one block, repeated twice, in its block diagonalization. If $n$ equals 2, then $\mathscr{A}_{n-2}$ consists of scalar multiples of the identity, contradicting $J(\mathscr{A})$ equals $n-1$. For $n$ bigger than 2, $\mathscr{A}_{n-3}$ must have an even number of blocks, yet $\#\mathscr{A}_{n-3}$ is no bigger than 3. Hence $\#\mathscr{A}_{n-3}$ is also 2 and $\mathscr{A}_{n-3}$ equals $\mathscr{A}_{n-2}$ which would imply that $J(\mathscr{A})$ is less than $n-2$.

COROLLARY 2.13. *If $\mathscr{A}$ has no multiplicity in its block diagonalization, then $J(\mathscr{A})$ is less than or equal to* 1.

*Proof.* Let $P_1, ..., P_r$ be the disjoint irreducible projections in the center of $\mathscr{A}_1$. Then the $P_i$'s are in the center of $\mathscr{A}_0$ as well since $\mathscr{A}_0$ has no multiplicity. Thus $X_i$ is the sum of the $P_j X_i P_j$'s so $X_i$ is in $\mathscr{A}_1$.

**2.14.** We now use these results on algebras formed by taking commutators to analyze $\mathscr{A}_j^{\mathscr{K}}(z)$, the algebra generated by the covariant derivatives of the curvature of order at most $j$.

DEFINITION 2.14.1. The $j$th coalescing set for the curvature, denoted $\mathscr{C}_j$, is the set of all $z$ in $\Omega$ such that dim $\mathscr{A}_i^{\mathscr{K}}(z)$ fails to be locally constant for at least one $i$, $0 \leqslant i \leqslant j$.

The $j$th coalescing set is closed and nowhere dense in $\Omega$.

DEFINITION 2.14.2. Let $E$ be a $C^\infty$ Hermitian vector bundle over $\Omega$ an open subset of $\mathbb{C}^k$ , with metric-preserving connection $D$. Then $E$ is a *multiplicity-free bundle* if $\mathscr{A}_0^{\mathscr{K}}(z)$, the algebra generated by the curvatures themselves and by the identity, has no multiplicity for all $z$ not in the zeroth coalescing set. (No multiplicity means that if $s$ is an orthonormal frame in a neighborhood of $z$ in which $\mathscr{A}_0(\mathscr{K}, s)(z)$ has a block diagonalization, then it has no multiplicity.)

DEFINITION 2.14.3. For $E$ as in the definition above, we define the *generating order* of the connection $D$, denoted by $g(D, \Omega)$, to be the smallest integer $j$ such that $\mathscr{A}^{\mathscr{K}}(z)$ (the algebra of all covariant derivatives of curvature) is generated by covariant derivatives of the curvature of total order at most $j$, for all $z$ in $\Omega$ minus the $j$th coalescing set.

Note that no finite order need suffice to generate $\mathscr{A}^{\mathscr{K}}(z)$ for *all* $z$ in $\Omega$, since the curvature and its first $N$ covariant derivatives could all be zero at some fixed $z_0$ in $\Omega$. If $\tilde{\Omega}$ is contained in $\Omega$, $\tilde{\Omega}$ open, then it is possible to have $g(D, \tilde{\Omega})$ strictly less than $g(D, \Omega)$.

We now come to the important concept of a generic bundle. We will show in the Appendix that these really are generic, in the sense that they form an open dense set in the set of all bundles. The definition splits naturally into two cases, depending on the dimension of the base space.

DEFINITION 2.14.4. Let $E$ be a $C^\infty$ Hermitian vector bundle with metric-preserving connection $D$, over $\Omega$ an open subset of $\mathbb{C}^k$. Then $E$ is a *generic bundle* if:

(i) $E$ is multiplicity-free (2.14.2), when $k$ equals 1, or

(ii) the generating order $g(D, \Omega)$ equals zero (the curvatures themselves generate everything off the zeroth coalescing set), when $k$ is bigger than 1.

Note that when $k$ is 1, condition (ii) is never satisfied (unless the bundle is one dimensional), since there is just one matrix $\mathscr{K}^{p;q}_{i;j}$, namely, $\mathscr{K}^{1;1}_{1;1}$.

Multiplicity-free bundles are particularly easy to analyze, since the generating order turns out to be at most 1. The example in 2.6 is non-generic for $n$ greater than 2. An open question is whether there exist non-trivial examples of non-generic bundles arising in a natural way from operator theory or from complex variables (cf. [CD1, 4.43]).

We now use the results on algebras formed by commutators to show the main result of this section.

PROPOSITION 2.15. *Let $E$ be an $n$-dimensional $C^\infty$ Hermitian vector bundle over $\Omega$ an open subset of $\mathbb{C}^k$, with metric-preserving connection $D$. Then the generating order is less than $n$, that is,*

$$0 \leqslant g(D, \Omega) \leqslant n-1 \tag{2.15.1}$$

*and is at most 1 if $E$ is multiplicity-free.*

*Proof.* Let $\mathscr{J}$ be the first integer $j$ such that $\mathscr{A}^{\mathscr{K}}_{j+1}(z)$ equals $\mathscr{A}^{\mathscr{K}}_j(z)$ for all $z$ in $\Omega$ minus the $j$th coalescing set $\mathscr{C}_j$. On each component of $\Omega$ minus $\mathscr{C}_{n^2}$, the dimensions of $\mathscr{A}^{\mathscr{K}}_0(z), \ldots, \mathscr{A}^{\mathscr{K}}_{n^2}(z)$ are constant, so $\mathscr{A}^{\mathscr{K}}_l(z)$ equals $\mathscr{A}^{\mathscr{K}}_{l+1}(z)$ for some $l$ not greater than $n^2$, for all $z$ in the component. Thus $\mathscr{A}^{\mathscr{K}}_{l+2}(z)$ equals $\mathscr{A}^{\mathscr{K}}_{l+1}(z)$ as well by the Leibnitz rule for covariant derivatives (2.3.1.), so we can show that $\mathscr{A}^{\mathscr{K}}_{n^2+1}(z)$ equals $\mathscr{A}^{\mathscr{K}}_{n^2}(z)$ for all $z$ in the component. Thus $\mathscr{T}$ is at most $n^2$.

Let $\Omega_0$ be a connected open subset of $\Omega$ minus $\mathscr{C}_{\mathscr{T}}$, on which there is an orthonormal frame $s$ such that $\mathscr{A}_0(\mathscr{K}, s)(z),\ldots, \mathscr{A}_{\mathscr{T}}(\mathscr{K}, s)(z)$ are simultaneously $C^\infty$ block diagonal for all $z$ in $\Omega_0$. The existence of a cover of $\Omega$ minus the $\mathscr{T}$th coalescing set by such $\Omega_0$'s is guaranteed by Corollary I (cf. (2.5.1)).

The matrix of connection 1-forms can be written

$$\Theta(s) = \sum_{i=1}^{k} \Theta_i'(s)\, dz_i + \sum_{i=1}^{k} \Theta_i''(s)\, d\bar{z}_i \tag{2.15.2}$$

for $C^\infty$ $n \times n$ matrix-valued functions $\Theta_i'(s)$ and $\Theta_i''(s)$, for $i$ equal to $1,\ldots, k$. Since $\Theta(s)$ is skew-symmetric,

$$(\Theta_i'(s))^* = -\Theta_i''(s). \tag{2.15.3}$$

If $\chi: E \to E$ is a bundle map, $\chi(s)$ its matrix with respect to $s$, then by 2.3.5 and by Definition 2.3,

$$\chi_{z_i}(s) = [\Theta_i'(s), \chi(s)] + \frac{d\chi(s)}{dz_i}$$

and

$$\chi_{\bar{z}_i}(s) = [\Theta_i''(s), \chi(s)] + \frac{d\chi(s)}{d\bar{z}_i}. \tag{2.15.4}$$

But if $\chi$ is in $\Gamma_j^{\mathscr{K}}$ for some $j$, $0 \leqslant j \leqslant \mathscr{T}$, then $\chi(s)$ is in $\mathscr{A}_j(\mathscr{K}, s)(z)$. Thus $d\chi(s)/dz_i$ and $d\chi(s)/d\bar{z}_i$ are in $\mathscr{A}_j(\mathscr{K}, s)(z)$ for all $i$. Therefore $\mathscr{A}_{j+1}(\mathscr{K}, s)(z)$ is formed by taking the algebra generated by $\mathscr{A}_j(\mathscr{K}, s)(z)$ and commutators with the $\Theta_i'(s)$ and $\Theta_i''(s)$, for all $j$ not greater than $\mathscr{T}$, and all $z$ in $\Omega_0$. By Proposition 2.11, and Corollary 2.13, $\mathscr{T}$ is less than $n$ and is at most 1 if $E$ is multiplicity-free. In particular, $\mathscr{A}_l(\mathscr{K}, s)(z)$ equals $\mathscr{A}_{\mathscr{T}}(\mathscr{K}, s)(z)$ for all $l$ greater than $\mathscr{T}$ and for all $z$ in $\Omega_0$, which shows that $\mathscr{T}$ is the generating order $g(D, \Omega)$.

The example in 2.6 shows that the Proposition is sharp.

**2.16.** It does not seem to simplify any of the proofs if $E$ is a real-analytic bundle with real-analytic connection $D$. The $j$th coalescing sets can still be non-empty. There is one simplification, however:

PROPOSITION 2.17. *Let $E$ be an $n$-dimensional, real-analytic, Hermitian vector bundle over $\Omega$ open in $\mathbb{C}^k$. If $E$ has a metric-preserving, real-analytic connection $D$, then the coalescing set of the curvature is empty.*

*Proof.* Fix $z_0$ in $\Omega$ and let $s$ be a real-analytic frame for $E$ over a neighborhood of $z_0$. Then $\Theta(s)$, the $\mathscr{K}_{ij}^{p,q}(s)$, and all their covariant derivatives are real-analytic. Let $Q_1(z),..., Q_l(z)$ be polynomials in the covariant derivatives of the curvature (with constant coefficients) such that $Q_1(z_0),..., Q_l(z_0)$ are a basis for $\mathscr{A}^{\mathscr{K}}(z_0)$.

Denote by $e_{ij}$ the real-analytic bundle map of $E$ into itself (over the neighborhood of $z_0$) where $e_{ij}(s_j)$ equals $s_i$ and $e_{ij}(s_k)$ is 0 for $k$ not equal to $j$. Then there exist $n^2 - l$ of the $e_{ij}$'s such that together with the $Q_i$'s they form a basis at $z_0$ for all linear transformations of $E_{z_0}$ into itself, and hence for $E_z$ into itself at any $z$ near $z_0$. Denote these $e_{ij}$'s by $R_1,..., R_t$, where $t$ is $n^2 - l$.

We wish to show that $\mathscr{A}^{\mathscr{K}}(z)$ is $l$-dimensional for all $z$ near $z_0$. Let $Q_0$ be in $\mathscr{A}^{\mathscr{K}}(z')$ for $z'$ near $z_0$. Then there exists $Q$ a polynomial in the covariant derivatives of curvature, with constant coefficients, such that $Q(z')$ is just $Q_0$. Now $Q(z)$ equals $\sum a_i(z)\, Q_i(z) + \sum b_j(z)\, R_j(z)$ where the $a_i$'s and $b_j$'s are real-analytic. Let $\tilde{Q}$ be $Q - \sum a_i Q_i$. Then $\tilde{Q}(z)$ equals $\sum b_j(z)\, R_j(z)$. Furthermore, at $z_0$, $\tilde{Q}$ and all its covariant derivatives to all orders are in $\mathscr{A}^{\mathscr{K}}(z_0)$. Thus $b_j(z_0) = 0$ for all $j$; and since $\tilde{Q}_{z_i} = \sum (\partial b_j/\partial z_i)\, R_j + \sum b_j(R_j)_{z_i}$ we see that $(\partial b_j/\partial z_i)(z_0)$ is zero also. By induction, all partial derivatives (with respect to $z_i$ and $\bar{z}_j$) of the $b_j$'s are 0 at $z_0$. Thus the $b_j$'s are identically zero at $z_0$ and hence $Q$ equals $\sum a_i Q_i$. But then $Q_0$ is in the span of $Q_1(z'),..., Q_l(z')$ so $\mathscr{A}^{\mathscr{K}}(z')$ is $l$-dimensional.

## 3. Equivalence of Connections

**3.1.** We apply the results on Diagonalization of Connections to determine when two connections are equivalent by means of comparing covariant derivatives of the curvatures. We show first: if at each point the curvatures and their covariant derivatives match up to high enough order, then the connections are (locally) equivalent. We then show, that off a closed nowhere dense subset $\mathscr{C}$, the order necessary is just the dimension of the bundles; indeed order $g(D, \Omega) + 1$ will work (cf. Proposition 2.15). A continuity argument then gives that this order suffices at every point in $\Omega$ minus the coalescing set of the curvature (2.4.1).

**3.2.** Let $E$ and $\tilde{E}$ be $n$-dimensional $C^\infty$ Hermitian vector bundles with metric-preserving connections $D$ and $\tilde{D}$ over $\Omega$ an open subset of $\mathbb{C}^k$. If $\chi$ is in $\Gamma^{\mathscr{K}}$ we denote by $\tilde{\chi}$ the corresponding element in $\Gamma^{\tilde{\mathscr{K}}}$ where $\tilde{\mathscr{K}}$ refers to the curvatures of $\tilde{D}$ for $\tilde{E}$; e.g., if $\chi$ is $(\mathscr{K}_{ij}^{1,1})_{z_l \bar{z}_j}$ then $\tilde{\chi}$ is $(\tilde{\mathscr{K}}_{ij}^{1,1})_{z_l \bar{z}_j}$.

LEMMA 3.3. *For fixed j, let $z_0$ be a point in $\Omega$ minus the jth coalescing set such that for each z close enough to $z_0$ there is an isometry $\varphi_z : E_z \to \tilde{E}_z$ which* intertwines covariant derivatives of curvature to order j, *that is,*

$$\varphi_z \circ \chi|_z = \tilde{\chi}|_z \circ \varphi_z \tag{3.3.1}$$

*for all $\chi$ in $\Gamma_j^{\mathscr{H}}$. (Note that $\varphi_z$ need not even be continuous in z.) Then there exists a neighborhood $\Omega_0$ of $z_0$ and a $C^\infty$ isometric bundle map $\Phi: E|_{\Omega_0} \to \tilde{E}|_{\Omega_0}$ such that*

$$\Phi \circ \chi = \tilde{\chi} \circ \Phi \qquad on \quad \Omega_0 \tag{3.3.2.}$$

*for all $\chi$ in $\Gamma_j^{\mathscr{H}}$, and $\Phi(z_0) = \varphi_{z_0}$.*

*Thus pointwise unitary equivalence of the $\chi$'s and $\tilde{\chi}$'s can be achieved smoothly.*

*Proof.* Let $s$ and $\tilde{s}$ be orthonormal frames for $E$ and $\tilde{E}$ which give $C^\infty$ diagonalizations of $\mathscr{A}_j(\mathscr{K}, s)(z)$ and $\mathscr{A}_j(\tilde{\mathscr{K}}, \tilde{s})(z)$ near $z_0$, and denote by $U_z$ the matrix of $\varphi_z$ relative to $s$ and $\tilde{s}$. We define a $*$-isomorphism $\psi(z)$ from $\mathscr{A}_j(\mathscr{K}, s)(z)$ to $\mathscr{A}_j(\tilde{\mathscr{K}}, \tilde{s})(z)$, for $z$ close enough to $z_0$, by

$$\psi(z)(X) = U_z X U_z^{-1} \tag{3.3.3}$$

for all $X$ in $\mathscr{A}_j(\mathscr{K}, s)(z)$.

By (3.3.2) we have

$$\psi(z)(\chi(s)) = \tilde{\chi}(\tilde{s}) \tag{3.3.4}$$

for all $\chi$ in $\Gamma_j^{\mathscr{H}}$. Thus $\psi$ gives an equivalence in the sense of Definition 1.15.1 and $\psi$ is $C^\infty$; the rank condition (1.15.3) is trivial by (3.3.1). Furthermore, by Proposition 1.16, there exists a $C^\infty$ unitary matrix function $U(z)$ such that on a small enough neighborhood $\Omega_0$ of $z_0$ we have

$$U(z)\,\chi(s)\,U(z)^{-1} = \tilde{\chi}(\tilde{s}). \tag{3.3.5}$$

Let $\Phi(z)$ be the bundle map from $E|_{\Omega_0}$ to $\tilde{E}|_{\Omega_0}$ which has $U(z)\,U(z_0)^{-1}\,U_{z_0}$ as its matrix. Since $U(z_0)^{-1}\,U_{z_0}$ commutes with $\chi(s)|_{z_0}$ for all $\chi$ in $\Gamma_j^{\mathscr{H}}$, it is in the commutant of $\mathscr{A}_j(\mathscr{K}, s)(z_0)$ for all $z$ near $z_0$. Thus $\Phi$ satisfies (3.3.2).

DEFINITION 3.4. If $\chi$ is in $\Gamma^{\mathscr{H}}$, the *total bi-order* of $\chi$ is $(p, q)$ if $\chi$ is a covariant derivative of the curvature with respect to $p$ of the $z_i$'s and $q$ of the $\bar{z}_j$'s.

Note that the total order is just $p + q$.

DEFINITION 3.5. Let $E$ and $\tilde{E}$ be Hermitian vector bundles over $\Omega$ open in $\mathbb{C}^k$, with metric-preserving connections $D$ and $\tilde{D}$. Let $j$ be a positive integer. Then $E$ and $\tilde{E}$ are *equivalent to order j* at a point $z$ in $\Omega$ if there exists an isometry $\varphi_z$ from $E_z$ to $\tilde{E}_z$ such that

$$\varphi_z \circ \chi|_z = \tilde{\chi}|_z \circ \varphi_z \tag{3.5.1}$$

for all $\chi$ of bi-order $(p, q)$ where

$$p+q \leqslant j \qquad \text{but } (p, q) \neq (0, j) \quad \text{or} \quad (j, 0). \tag{3.5.2}$$

$E$ and $\tilde{E}$ are equivalent to order $j$ on $\Omega$ if they are equivalent to order $j$ at each $z$ in $\Omega$ (with no assumption on how $\varphi_z$ varies with $z$).

Note that equivalence to order 1 means that the curvatures (but *not* any covariant derivatives) are intertwined. Equivalence to order 2 means that $\chi$, $\chi_{z_i}, \chi_{\bar{z}_j}$, and $\chi_{z_i\bar{z}_j}$ (but not $\chi_{z_i^2}$ or $\chi_{\bar{z}_j^2}$) are intertwined for all curvatures $\chi$ ($\chi$ is in $\Gamma_0^{\mathscr{V}}$).

The reason for excluding $(p, q)$ equal to $(0, j)$ or $(j, 0)$ arises from applications to Hermitian holomorphic vector bundles induced in a natural way from operators with an open set of eigenvalues or from holomorphic curves in Grassmann manifolds (cf. [CD1]).

**3.6.** There exist a finite number of invariant functions which determine equivalence to order $j$. This follows from applying the results of Specht [S] and Pearcy [P] at each $z$ in $\Omega$. We give their proof of this result, generalized to the case of several $n \times n$ matrices.

PROPOSITION 3.7. *Let* $\{A_1, ..., A_l\}$ *and* $\{B_1, ..., B_l\}$ *be self-adjoint collections of* $n \times n$ *matrices. If*

$$\operatorname{tr} A_{i_1}, ..., A_{i_m} = \operatorname{tr} B_{i_1}, ..., B_{i_m} \tag{3.7.1}$$

*for all* $1 \leqslant i_1, ..., i_m \leqslant l$ *and all* $1 \leqslant m \leqslant 2n^2 - 2$ *then* $A_1, ..., A_l$ *and* $B_1, ..., B_l$ *are simultaneously unitarily equivalent.*

*Thus there are at most* $l^{2n^2}$ *invariants necessary for simultaneous unitary equivalence, for* $l$ *greater than* 1, *and there are* $n$ *invariants* ($\operatorname{tr} A_1^j$, $1 \leqslant j \leqslant n$) *when* $l$ *is* 1.

*Proof.* Let $\mathscr{P}_i$ be the space of all polynomials of weight no more than $i$ in $l$ non-commuting $n \times n$ matrix variables. If $P$ is in $\mathscr{P}_i$, then we denote by $P(A)$ the matrix $P(A_1, ..., A_l)$.

Let $\mathscr{A}_i$ denote the subspace of $n \times n$ matrices consisting of the $P(A)$'s for all $P$ in $\mathscr{P}_i$. If $\mathscr{A}_{i+1}$ equals $\mathscr{A}_i$ for some $i$, then $\mathscr{A}_i$ equals $\mathscr{A}$, the algebra generated by $A_1, ..., A_l$ and the identity. If $\dim \mathscr{A}_1$ is greater than 2, then $\mathscr{A}_{n^2-2}$ equals $\mathscr{A}$ by consideration of the chain $\mathscr{A}_1 \subset \cdots \subset \mathscr{A}_{n^2-1}$. If $\dim \mathscr{A}_1$ is no more than 2, then all the $A_i$'s are linear combinations of the identity and of, say, $A_1$. But then $A_1^*$ is also such a combination, so $A_1$ is normal and $\dim \mathscr{A}$ is at most $n$. So again $\mathscr{A}_{n^2-2}$ equals $\mathscr{A}$.

We define a $*$-isomorphism $\Phi$ from $\mathscr{A}$ to $\mathscr{B}$, the algebra generated by $B_1, ..., B_l$ by

$$\Phi(P(A)) = P(B). \tag{3.7.2}$$

for all $P$ in $\mathscr{P}_{n^2-2}$. We need to show that $\Phi$ is well defined and that $\Phi(A_j P(A)) = B_j P(B)$ for $j = 1,\dots, l$ and all $P$ in $\mathscr{P}_{n^2-2}$. Thus it suffices to show that (3.7.2) is well defined for all $P$ in $\mathscr{P}_{n^2-1}$, or equivalently we need to show that if $P(A)$ equals zero for $P$ in $\mathscr{P}_{n^2-1}$, then $P(B)$ is zero. But a matrix $X$ is 0 if and only if $\operatorname{tr} XX^*$ is 0. So it suffices to show that $\operatorname{tr} P(A)P(A)^*$ equals $\operatorname{tr} P(B)P(B)^*$ for all $P$ in $\mathscr{P}_{n^2-1}$ or equivalently, that $\operatorname{tr} P(A)$ equals $\operatorname{tr} P(B)$ for all $P$ in $\mathscr{P}_{2n^2-2}$ which is exactly the condition (3.7.1).

Since $\Phi$ is a $*$-isomorphism (there is an obvious inverse), and $\Phi$ preserves the trace and hence the rank of projections, then $\Phi$ is spatial (Proposition 1.16), that is, there is a unitary $U$ such that $\Phi(P(A)) = UP(A)U^{-1}$. In particular, $UA_iU^{-1}$ is $B_i$ for all $i$.

The (crude) bound on the number of conditions in (3.7.1) comes from summing the geometric series $1 + l + \cdots + l^{2n^2-2}$. Pearcy [P] points out that the bound is far from optimal. When $l$ is 2 and $n$ is 3 the sharp bound is 7, not of order $10^5$.

Thus we obtain the following condition for equivalence of bundles:

PROPOSITION 3.8. *Let $E$ and $\tilde{E}$ be Hermitian vector bundles of dimension $n$ over $\Omega$ open in $\mathbb{C}^k$, with metric-preserving connections $D$ and $\tilde{D}$. Then $E$ and $\tilde{E}$ are equivalent to order $j$ on $\Omega$ (Definition 3.5) if and only if*

$$\operatorname{tr} \chi_1 \cdots \chi_m = \operatorname{tr} \tilde{\chi}_1 \cdots \tilde{\chi}_m \qquad on\ \Omega \tag{3.8.1}$$

*for all $\chi_1,\dots,\chi_m$ in $\Gamma_j^{\mathscr{K}}$ (with bi-degree $\chi_i$ not equal to $(0,j)$ or $(j,0)$) and for all $m$ from 1 to $2n^2-2$.*

*Proof.* Since $\Gamma_j^{\mathscr{K}}$ is self-adjoint by (2.3.7), the result follows from the previous Proposition.

Note that $\Gamma_j^{\mathscr{K}}$ consists of a finite number of bundle maps. Thus the number of traces necessary is bounded in terms of the fibre dimension $n$ and the base dimension $k$.

The following Lemma is a special case of the Equivalence Theorem and the main step in its proof.

LEMMA 3.9. *Let $E$ and $\tilde{E}$ be $n$-dimensional Hermitian vector bundles over $\Omega$ an open subset of $\mathbb{C}^k$, with metric-preserving connections $D$ and $\tilde{D}$. Let $z_0$ be in $\Omega$, $z_0$ not in the coalescing set for the curvature of $E$. Let $\Gamma_j^{\mathscr{K}}(z_0)$ generate $\mathscr{A}^{\mathscr{K}}(z_0)$, where $\mathscr{K}$ refers to the curvature of $D$. If for each $z$ in a neighborhood of $z_0$ there is an isometry $\varphi_z$ from $E_z$ to $\tilde{E}_z$ which intertwines covariant derivatives of the curvature of order at most $j+1$, then $E$ and $\tilde{E}$ are equivalent in a neighborhood of $z_0$. Furthermore, the equivalence $\Phi$ from $E$ to $\tilde{E}$ can be chosen so that $\Phi|_{z_0}$ equals $\varphi_{z_0}$.*

*Proof.* By Lemma 3.3, there exists a $C^\infty$ isometry $\varphi$ from $E$ to $\tilde{E}$ on a small enough neighborhood $\Omega_0$ of $z_0$, such that $\varphi \circ \chi = \tilde{\chi} \circ \varphi$ on $\Omega_0$ for all $\chi$ a covariant derivative of curvature with order at most $j+1$, and $\varphi|_{z_0}$ equals $\varphi_{z_0}$.

By Proposition 2.5, we can assume that $\Omega_0$ is small enough so that there is a $C^\infty$ orthonormal frame $s$ on $\Omega_0$ with

$$\mathscr{A}(\mathscr{H}, s)(z) \equiv M(\mathscr{N}, \otimes \mathscr{M}) \qquad \text{on} \quad \Omega_0 \tag{3.9.1}$$

and

$$\Theta(s) = (\Theta_1, \dots, \Theta_r) \otimes I_{\mathscr{M}}. \tag{3.9.2}$$

Let $\tilde{s}$ be the orthonormal frame for $\tilde{E}$ on $\Omega_0$ obtained by applying the isometry $\varphi$ to the frame $s$. Then

$$\chi(s) = \tilde{\chi}(\tilde{s}) \tag{3.9.3}$$

for all $\chi$ in $\Gamma_{j+1}^{\mathscr{H}}$. Let $\delta$ be the $n \times n$ matrix of 1-forms

$$\delta = \Theta(s) - \tilde{\Theta}(\tilde{s}), \tag{3.9.4}$$

the difference of the matrices of connection 1-forms.

For any $\chi$ in $\Gamma_j^{\mathscr{H}}$,

$$\begin{aligned}
[\delta, \chi(s)] &= (D\chi - \chi D)(s) - d\chi(s) - \{(\tilde{D}\tilde{\chi} - \tilde{\chi}\tilde{D})(\tilde{s}) - d\tilde{\chi}(\tilde{s})\} \\
&= \sum_i (\chi_{z_i} dz_i + \chi_{\bar{z}_i} d\bar{z}_i)(s) - \sum_i (\tilde{\chi}_{z_i} dz_i + \tilde{\chi}_{\bar{z}_i} d\bar{z}_i)(\tilde{s}) \\
&= 0
\end{aligned} \tag{3.9.5}$$

by (0.4.2), Definition 2.3, and (3.9.3), since the $\chi_{z_i}$'s and $\chi_{\bar{z}_i}$'s are of order at most $j+1$. Since $\Gamma_j^{\mathscr{H}}(z)$ generates $\mathscr{A}^{\mathscr{H}}(z)$ for $z$ close enough to $z_0$, we may further shrink $\Omega_0$ and obtain from (3.9.5) that $\delta$ commutes with $M(\mathscr{N}, \otimes \mathscr{M})$ on $\Omega_0$. By (3.9.2), then $\delta$ anticommutes with $\Theta(s)$ (*anti*-commutes since $\Theta(s)$ and $\delta$ are matrices of 1-forms). Furthermore,

$$\begin{aligned}
d\delta &= K(s) - \Theta(s) \wedge \Theta(s) - \tilde{K}(\tilde{s}) + \tilde{\Theta}(\tilde{s}) \wedge \tilde{\Theta}(\tilde{s}) \qquad \text{(by (0.7.1))} \\
&= \tilde{\Theta}(\tilde{s}) \wedge \tilde{\Theta}(\tilde{s}) - \Theta(s) \wedge \Theta(s) \qquad \text{(by (3.9.3))} \\
&= -\delta \wedge \tilde{\Theta}(s) - \Theta(s) \wedge \delta \\
&= \delta \wedge \delta
\end{aligned} \tag{3.9.6}$$

by anti-commutativity of $\delta$ and $\Theta(s)$.

Since $\delta$ commutes with $M(\mathcal{N}, \otimes \mathcal{M})$, it has the form (cf. (2.5.3))

$$\delta = I_{n_1} \otimes \delta_1 + \cdots + I_{n_r} \otimes \delta_r \tag{3.9.7}$$

where $\delta_i$ is in $M(m_i, \mathbb{C})$ tensored with the 1-forms. By (3.9.6), we obtain

$$d\delta_i = \delta_i \wedge \delta_i \qquad \text{for } 1 \leqslant i \leqslant r. \tag{3.9.8}$$

Since $\Theta(s)$ and $\tilde{\Theta}(\tilde{s})$ are skew-adjoint, so are the $\delta_i$'s. By the Frobenius Integration Theorem, for each $i$ there exists a $C^\infty$ $m_i \times m_i$ unitary-valued matrix function $U_i$ such that $U_i$ is defined in a neighborhood of $z_0$, $U_i$ equals the identity at $z_0$, and (cf. [F, p. 102])

$$\delta_i = (dU_i)\, U_i^{-1}.$$

Thus if we let $U$ be the $C^\infty$ $n \times n$ unitary-valued matrix function on a neighborhood of $z_0$ defined by

$$U = I_{n_1} \otimes U_1 + \cdots + I_{n_r} \otimes U_r$$

then we have

$$U \text{ is the identity at } z_0, \tag{3.9.9}$$

$$\delta = (dU)\, U^{-1}, \tag{3.9.10}$$

and

$$U \text{ is in the commutant of } M(\mathcal{N}, \otimes \mathcal{M}). \tag{3.9.11}$$

Let $\Phi$ be the isometry of $E$ onto $\tilde{E}$ defined in a neighborhood of $z_0$ by setting the matrix of $\Phi$ relative to the frames $s$ and $\tilde{s}$ equal to $U$. Then $\Phi$ at $z_0$ is just $\varphi_{z_0}$. Relative to the frames $s$ and $\tilde{s}$, $\Phi D - \tilde{D}\Phi$ has matrix

$$\begin{aligned} U\Theta(s) - \tilde{\Theta}(\tilde{s})\, U - dU &= \Theta(s)\, U - \tilde{\Theta}(\tilde{s})\, U - \delta U \\ &= 0 \end{aligned} \tag{3.9.12}$$

by (3.9.11), (3.9.10), and (3.9.4). Thus $\Phi D$ equals $\tilde{D}\Phi$ so $\Phi$ is the desired equivalence.

**3.10.** We can now prove the main result of this paper, the Equivalence Theorem.

THEOREM II (Equivalence). *Let $E$ and $\tilde{E}$ be $n$-dimensional Hermitian vector bundles over $\Omega$ an open subset of $\mathbb{C}^k$, with metric-preserving connections $D$ and $\tilde{D}$, which are equivalent to order $n$ on $\Omega$. Then on an open dense subset of $\Omega$, the complement of the coalescing set for the curvature of $E$, the bundles are locally equivalent.*

*Furthermore, if $E$ is generic (Definition 2.14.4), then equivalence to order 2 is sufficient for local equivalence off the coalescing set. If $D$ is real-analytic, local equivalence holds on all of $\Omega$.*

*Proof.* Let $J$ be the generating order for the connection $D$, $J = g(D, \Omega)$, from Definition 2.14.3. Thus for each $z_0$ in $\Omega$ minus the $J$th coalescing set, $z_0$ is also in $\Omega$ minus the coalescing set for the curvature. By Lemma 3.9, if at each point $z$ in $\Omega$ there is an isometry from $E_z$ to $\tilde{E}_z$ which intertwines covariant derivatives of the curvature to order $J+1$, then for each $z_0$ in $\Omega$ minus the $J$th coalescing set there is an equivalence of $E$ and $\tilde{E}$ in a neighborhood of $z_0$. An equivalence intertwines covariant derivatives to *all* orders, since if $s$ is an orthonormal frame for $E$ and $\tilde{s}$ is the corresponding frame for $\tilde{E}$ under the equivalence, then $\Theta(s)$ equals $\tilde{\Theta}(\tilde{s})$ so $\chi(s)$ equals $\tilde{\chi}(s)$ for all $\chi$ in $\Gamma^{\mathscr{K}}$. Since $\Omega$ minus the $J$th coalescing set is dense in $\Omega$, Proposition 3.8 and continuity of the traces show that at each point in $\Omega$ there is an isometry which intertwines covariant derivatives to any order. Hence by Lemma 3.9 the bundles are locally equivalent on $\Omega$ minus the coalescing set, under the assumption that at each point there is an isometry intertwining covariant derivatives to order $g(D, \Omega) + 1$. If $g(D, \Omega)$ is less than $n-1$ (it is at most $n-1$ by Proposition 2.15) we have proved the first part of the Theorem, since equivalence to order $n$ implies intertwining of all covariant derivatives of order $n-1$ and hence *a fortiori* of order $g(D, \Omega) + 1$. Similarly, we are done if $E$ is generic and $g(D, \Omega)$ is 0.

We are left with two cases: $g(D, \Omega)$ is $n-1$ or $E$ is generic with $g(D, \Omega)$ equal to 1 (it is at most 1 by Proposition 2.15). In each of these cases, if $J$ is $g(D, \Omega)$ then $\mathscr{A}^{\mathscr{K}}_{J-1}(z)$ has no multiplicity for $z$ in $\Omega$ minus the $J$th coalescing set (by definition when $E$ is generic and by Lemma 2.12 when $J$ is $n-1$). Thus it suffices to show that if $\mathscr{A}^{\mathscr{K}}_{J-1}(z)$ has no multiplicity, then equivalence to order $J+1$, which doesn't a priori imply intertwining of *all* covariant derivatives to order $J+1$, is all that we really need to apply Lemma 3.9.

Since equivalence to order $J+1$ implies intertwining by an isometry of covariant derivatives to order $J$, (3.9.5) still implies that $\delta$ commutes with all $\chi(s)$ for $\chi$ in $\Gamma^{\mathscr{K}}_{J-1}$. Since $\mathscr{A}^{\mathscr{K}}_{J-1}(z)$ has no multiplicity, then $\delta$ is diagonal, not just block-diagonal. Decompose $\delta$ into $(1,0)$ and $(0,1)$ parts:

$$\delta = \delta' + \delta'' \tag{3.10.1}$$

where $\delta'$ is a matrix with $(1,0)$-form coefficients (that is, combinations of the $dz_i$'s) and $\delta''$ has $(0,1)$-form coefficients (combinations of the $d\bar{z}_j$'s). Then $\delta'$ and $\delta''$ are diagonal, and the skew-adjointness of $\delta$ implies

$$(\delta')^* = -\delta''. \tag{3.10.2}$$

Since $\delta'$ is diagonal, if $X$ is any $n \times n$ matrix, then $X$ commutes with $\delta'$ if and only if it commutes with $\delta''$.

Thus if $\chi$ is in $\Gamma_J^{\mathscr{V}}$, then $\chi(s)$ commutes with $\delta$ if and only if it commutes with either $\delta'$ or $\delta''$. But (3.9.5) implies

$$[\delta', \chi(s)] = \sum_i (\chi_{z_i}(s) - \tilde{\chi}_{z_i}(\tilde{s}))\, dz_i \tag{3.10.3}$$

and

$$[\delta'', \chi(s)] = \sum_j (\chi_{\bar{z}_j}(s) - \tilde{\chi}_{\bar{z}_j}(\tilde{s}))\, d\bar{z}_j \tag{3.10.4}$$

so $\chi(s)$ commutes with $\delta$ for $\chi$ in $\Gamma_J^{\mathscr{V}}$ if $\chi_{z_i}(s)$ equals $\tilde{\chi}_{z_i}(\tilde{s})$ for all $i$, or if $\chi_{\bar{z}_j}(s)$ equals $\tilde{\chi}_{\bar{z}_j}(\tilde{s})$ for all $j$, $1 \leqslant i, j \leqslant k$. Thus equivalence to order $J+1$ implies that $[\delta, \chi(s)]$ is zero for all $\chi$ in $\Gamma_J^{\mathscr{V}}$, since if $\chi$ has total bi-order $(J, 0)$, then $\chi_{\bar{z}_j}$ has total bi-order $(J, 1)$ and *is* intertwined by definition of equivalence to order $J+1$, so $\chi_{\bar{z}_j}(s)$ equals $\tilde{\chi}_{\bar{z}_j}(\tilde{s})$ can be arranged. Since $\delta$ then commutes with $\mathscr{A}_J(\mathscr{H}, s)(z)$, we can continue the proof of Lemma 3.9, and we are done with the first two parts of the Theorem.

The last part, on real-analytic connections, follows trivially from Proposition 2.17.

**3.11.** We now use the example in 2.6 to show that the Equivalence Theorem is sharp. By Proposition 2.7, the family of connections $D_\alpha$ all have the same covariant derivatives of curvature for total order less than $n$.

If $D_\alpha$ and $D_\beta$ are equivalent in a neighborhood of a point $z_0$ in $\mathbb{C}$, and if $U$ is the matrix of the equivalence relative to the given frame $s$, then $U$ is a $C^\infty n \times n$ unitary matrix-valued function for $z$ near $z_0$. Since an equivalence intertwines covariant derivatives of the curvature to all orders, then by (2.7.1) with $j$ equal to 1, $U(z)$ is in the commutant of $M(n, \mathbb{C})$ for all $z$ near $z_0$, so $U(z)$ is a scalar multiple of the identity, $U(z) = u(z)I$, where $|u|$ is identically 1. Since $U$ represents an equivalence, then $u\Theta_\beta$ equals $\Theta_\alpha u + (du)I$, which implies that $u(\Theta_\beta - \Theta_\alpha)$ is (du)$I$ and by (2.6.2),

$$(du)I = 2i \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \beta-\alpha \end{pmatrix} u\,dy. \tag{3.11.1}$$

Thus $D_\alpha$ is equivalent to $D_\beta$ only if $\alpha$ equals $\beta$, though they are equivalent to order $n-1$ for all $\alpha$ and $\beta$. The $D_\alpha$'s are real-analytic and indeed $D_\alpha$ is the canonical connection relative to a holomorphic bundle with real-analytic Hermitian metric [CD1, 3.23].

## 4. Global Equivalence

**4.1.** Our main theorem gives sufficient conditions for *local* equivalence off a closed nowhere dense set, the coalescing set. In this section we begin the study of *global* equivalence. For real-analytic bundles and connections this is completely straightforward. If the base space $\Omega$ is simply connected, then local equivalence implies global. In the $C^\infty$ case, the existence of the coalescing set is a complicating factor which is only partially understood.

Let $E$ and $\tilde{E}$ be $n$-dimensional $C^\infty$ Hermitian vector bundles, over $\Omega$ a complex manifold, with metric-preserving connections $D$ and $\tilde{D}$, respectively. Let $\mathscr{C}$ be the coalescing set for $E$ with the connection $D$, so $\mathscr{C}$ is a closed nowhere-dense subset of $\Omega$. We assume that the conclusion of the Equivalence Theorem holds, namely, that $E$ and $\tilde{E}$ are locally equivalent on $\Omega - \mathscr{C}$. We first show that they are globally equivalent on $\Omega - \mathscr{C}$ if $\Omega - \mathscr{C}$ is simply connected, as one would expect. We then wish to determine when $E$ and $\tilde{E}$ are equivalent on all of $\Omega$. This depends on the topology of $\Omega$, as is apparent, and on $\Omega - \mathscr{C}$, in a manner which is not so apparent. If $\Omega - \mathscr{C}$ is disconnected then there need not exist a global equivalence on $\Omega$. If $\Omega - \mathscr{C}$ is connected and $\mathscr{C}$ is not too pathological then the only obstruction to global equivalence seems to be the topology of $\Omega$. The set $\mathscr{C}$ gives no obstruction to the extension of an equivalence, which is quite surprising.

For a different approach to this sort of problem see [Mo].

**4.2.** We begin with an example showing how the connectivity of $\Omega - \mathscr{C}$ affects global equivalence.

Let $\Omega$ be a connected manifold (smooth or complex) and let $\mathscr{C}$ be a closed nowhere-dense subset of $\Omega$ with the following property:

> There exists a $C^\infty$ real 1-form $\omega$ on $\Omega$ which vanishes to infinite order on $\mathscr{C}$ and $d\omega$ is never 0 off $\mathscr{C}$. (Actually it suffices to have $d\omega$ never zero to infinite order off $\mathscr{C}$.) (4.2.1)

For example, if $\Omega$ is $\mathbb{C}$ and $\mathscr{C}$ is the $y$-axis, we could take $\omega$ to be $e^{-1/x^2}\,dy$.

Let $E$ be the trivial 2-dimensional Hermitian vector bundle, $E$ equals $\Omega \times \mathbb{C}^2$, and let $s_1, s_2$ be the orthonormal frame given by $s_i(x) \equiv e_i$ where $e_1, e_2$ are the standard orthonormal basis for $\mathbb{C}^2$. We define a connection $D$ on $E$ by

$$Ds_1 = i\omega s_1, \qquad Ds_2 = 0 \tag{4.2.2}$$

or equivalently by defining the matrix $\Theta$ of connection 1-forms relative to the frame $s_1, s_2$:

$$\Theta = i\begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.2.3}$$

Since $\Theta$ is skew-adjoint, $D$ is metric-preserving. Then the curvature matrix $K$ (0.7.1) is given by

$$K = i\begin{pmatrix} d\omega & 0 \\ 0 & 0 \end{pmatrix} \tag{4.2.4}$$

and the coalescing set of the curvature (2.4.1) is exactly $\mathscr{C}$.

Let $\Omega_0$ be any connected open subset of $\Omega$ on which there is a $C^\infty$ $2 \times 2$ unitary matrix-valued function $U$ with

$$U\Theta = \Theta U + dU. \tag{4.2.5}$$

Then by (0.7.2) we have

$$UK = KU \tag{4.2.6}$$

which implies that $U$ is diagonal on $\Omega_0 - \mathscr{C}$. But (4.2.5) then implies that $dU = 0$ on $\Omega_0 - \mathscr{C}$ and hence on $\Omega_0$. Thus we obtain from (4.2.5):

$$U \text{ is constant and diagonal on } \Omega_0. \tag{4.2.7}$$

Example 4.2.8. We utilize the example above to construct an example of two bundles which are locally equivalent on $\Omega - \mathscr{C}$ but not globally equivalent *if $\Omega - \mathscr{C}$ is not connected.* Namely, we let $\tilde{E}$ equal $E$ and define $\tilde{D}$ by setting:

$$\tilde{\Theta} = \Theta \text{ on } \bar{\Omega}_1 \qquad \text{and} \qquad \tilde{\Theta} = J\Theta J^{-1} \text{ on } \bar{\Omega}_2 \tag{4.2.9}$$

where $J = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\Omega - \mathscr{C}$ equals $\Omega_1 \cup \Omega_2$ with $\Omega_1$ and $\Omega_2$ open in $\Omega$ and non-empty. Note that $\tilde{\Theta}$ is $C^\infty$ and skew-adjoint on all of $\Omega$ since it vanishes to infinite order on $\mathscr{C}$.

Now $D$ and $\tilde{D}$ are equal, hence equivalent, on $\Omega_1$ and $D$ and $\tilde{D}$ are equivalent on $\Omega_2$. If they were equivalent on all of $\Omega$, there would be a $C^\infty$ $2 \times 2$ unitary matrix-valued function $U$ on $\Omega$ such that $U\tilde{\Theta} = \Theta U + dU$. Let $U_1$ equal $U$ on $\Omega_1$ and $U_2$ equal $UJ$ on $\Omega_2$. Then $U_i$ satisfies (4.2.5) on $\Omega_i$ and hence by (4.2.7) $U_i$ is constant on each component of $\Omega_i$ and $U_i$ is diagonal. Since this implies $dU$ is 0 on $\Omega_1$ and on $\Omega_2$, $dU$ is 0 on all of $\Omega$ and hence $U$ itself must be constant on all of $\Omega$. This contradicts $UJ$ being diagonal on $\Omega_2$.

Example 4.2.9. With a slight modification we can arrange for $E$ and $\tilde{E}$ to be locally equivalent on all of $\Omega$ (instead of on $\Omega - \mathscr{C}$) but not globally equivalent. Here we assume that $\Omega - \mathscr{C}$ has three components $\Omega_1, \Omega_2$, and $\Omega_3$ where $\bar{\Omega}_1$ and $\bar{\Omega}_2$ are disjoint. We let $\Theta$ and $\tilde{\Theta}$ be as in (4.2.9) and set them both equal to 0 on $\Omega_3$. Then it is easy to see that $D$ and $\tilde{D}$ are equal on

$\Omega - \bar{\Omega}_2$ and equivalent on $\Omega - \bar{\Omega}_1$ so they are locally equivalent on all of $\Omega$. The same argument used in Example 4.2.8 shows that there is no global equivalence.

A particular instance of this example is when $\Omega$ is $\mathbb{C}^k$ and $\mathscr{C}$ is the set $\{z \in \mathbb{C}^k | \operatorname{Re} z_1 = \pm 1\}$. The 1-form $\omega$ is $\exp\{-(x_1^2 - 1)^{-2}\}\, dy_1$. Then $d\omega$ is non-zero on $\Omega_1 = \{z | \operatorname{Re} z_1 > 1\}$ and on $\Omega_2 = \{z | \operatorname{Re} z_1 < -1\}$, which is all we need.

This shows that local equivalence everywhere *even when $\Omega$ is contractible* does not necessarily imply global equivalence.

**4.3.** We have a few more observations to make before beginning to study global equivalence in earnest. The first is that equivalences are uniquely determined by their initial values.

PROPOSITION 4.4 (Uniqueness). *Let $E$ and $\tilde{E}$ be $n$-dimensional Hermitian vector bundles, over $\Omega$ a connected manifold, with metric-preserving connections $D$ and $\tilde{D}$. If $\Phi_1$ and $\Phi_2$ from $E$ to $\tilde{E}$ are equivalences with $\Phi_1$ equal to $\Phi_2$ at some point $z_0$ in $\Omega$, then $\Phi_1$ and $\Phi_2$ are identically equal on $\Omega$.*

*Proof.* The set where $\Phi_1$ and $\Phi_2$ are equal is closed. To see that it is open it suffices to assume that $\Omega$ is an open set contained in $\mathbb{R}^k$. Let $s$ be an orthonormal frame for $E$ in a neighborhood of $z_0$ and let $\Theta$ be the matrix of connection 1-forms for $D$. Similarly we have $\tilde{s}$ and $\tilde{\Theta}$ for $\tilde{E}$ and $\tilde{D}$. Let $A$ be the matrix of $\Phi_1 - \Phi_2$ relative to $s$ and $\tilde{s}$. Then by (0.4.11) we have

$$A\tilde{\Theta} = \Theta A + dA. \tag{4.4.1}$$

Restricting (4.4.1) to any line through $z_0$ gives a system of ordinary differential equations with a solution which is zero at $z_0$ and hence on the whole line. Thus $A$ is identically zero so $\Phi_1 \equiv \Phi_2$ in a neighborhood of $z_0$. By connectivity of $\Omega$, $\Phi_1 \equiv \Phi_2$ on all of $\Omega$.

Our second observation is a semi-global result which we will use shortly to obtain the expected global results.

LEMMA 4.5. *Let $E$ and $\tilde{E}$ be Hermitian vector bundles over $\Omega$ with metric-preserving connections $D$ and $\tilde{D}$. If $E$ and $\tilde{E}$ are locally equivalent on the complement of the coalescing set $\mathscr{C}$, then for each $z_0$ in $\Omega - \mathscr{C}$ there is a neighborhood $\Omega_0$ contained in $\Omega - \mathscr{C}$ with the following property:*

> *For each $z_1$ in $\Omega_0$, any equivalence of $E$ and $\tilde{E}$ in a connected neighborhood of $z_1$ in $\Omega_0$ extends to an equivalence on all of $\Omega_0$.* (4.5.1)

*Note.* Example 4.2.9 shows that the Lemma need not hold if we replace

$\Omega - \mathscr{C}$ by $\Omega$. If $\Omega$ is $\mathbb{C}^k$ there are equivalences on $|\operatorname{Re} z_1| < 1$ which do not extend to $|\operatorname{Re} z_1| \leqslant 1$.

*Proof.* Take $\Omega_0$ small enough so that there exists $\Phi_0 \colon E|_{\Omega_0} \to \tilde{E}|_{\Omega_0}$, an equivalence, and such that Proposition 2.5 (Diagonalization of Connections) holds, that is, there exists an orthonormal frame $s$ for $E$ with

$$\mathscr{A}(\mathscr{K}, s)(z) = M(\mathscr{N}, \otimes \mathscr{M}) \qquad \text{for all } z \text{ in } \Omega_0, \tag{4.5.2}$$

and

$$\Theta(s) = (\Theta_1, ..., \Theta_r) \otimes I_{\mathscr{M}} \tag{4.5.3}$$

where $\Theta$ is the matrix of connection forms of $D$ relative to $s$. Let the frame $\tilde{s}$ for $\tilde{E}$ be the image of $s$ under $\Phi_0$. Then $\tilde{\Theta}(\tilde{s}) = \Theta(s)$ since $\Phi_0$ is connection-preserving.

Let $U$ be the matrix of an equivalence on a connected neighborhood $\Omega_1$ contained in $\Omega_0$. Since the covariant derivatives of the curvatures for $D$ and $\tilde{D}$ have equal matrices relative to the frames $s$ and $\tilde{s}$, $U$ is in the commutant of $M(\mathscr{N}, \otimes \mathscr{M})$ and hence commutes with $\Theta(s)$. But by (0.4.4), $U\tilde{\Theta} = \Theta U + dU$, so $U$ is constant on $\Omega_1$. By extending $U$ to a constant on all of $\Omega_0$, we will still satisfy (0.4.4), and hence we can extend the equivalence to all of $\Omega_0$.

**4.6.** Let $(\Phi_0, \Omega_0)$ be a pair consisting of an equivalence $\Phi_0$ and an open subset $\Omega_0$ of $\Omega$, where $\Phi_0$ maps $E|_{\Omega_0}$ onto $\tilde{E}|_{\Omega_0}$. We put an equivalence relation $\sim$ on the collection of all such pairs relative to a fixed point $z_0$ in $\Omega$ by $(\Phi_0, \Omega_0) \sim (\Phi_1, \Omega_1)$ if $\Phi_0$ equals $\Phi_1$ in a neighborhood of $z_0$. The *germ* of $\Phi_0$, denoted $[\Phi_0]_{z_0}$, is the equivalence class of $\Phi_0$. We denote by $\operatorname{Equiv}(\Omega, E, \tilde{E})$ the set of all such germs at all points of $\Omega$ and give this the topology determined by taking all sets of the form $\{[\Phi_0]_z|_z \in \Omega_0\}$ as a basis. Then $\operatorname{Equiv}(\Omega, E, \tilde{E})$ is a sheaf of (non-abelian) groups with projection $\pi$ from $\operatorname{Equiv}(\Omega, E, \tilde{E})$ into $\Omega$ given by $\pi([\Phi_0]_z) = z$.

*Note.* Of course the sheaf depends on the choice of connections $D$ and $\tilde{D}$ but we omit this from the notation.

$\operatorname{Equiv}(\Omega, E, \tilde{E})$ is always Hausdorff by Proposition 4.4. It is not generally a covering space of $\Omega$, but restricted to $\Omega - \mathscr{C}$ it is, and this gives us our first, non-surprising, global result.

PROPOSITION 4.7. *Let $E$ and $\tilde{E}$ be Hermitian vector bundles over $\Omega$ with metric-preserving connections $D$ and $\tilde{D}$ which are locally equivalent on the complement of the coalescing set $\mathscr{C}$. If $\Omega - \mathscr{C}$ is simply connected, then $E$ and $\tilde{E}$ are equivalent on all of $\Omega - \mathscr{C}$.*

*Note.* $\mathscr{C}$ is empty if the bundles and connections are real-analytic (Proposition 2.17).

*Proof.* By Lemma 4.5, $\pi$ from Equiv($\Omega-\mathscr{C}$, $E$, $\tilde{E}$) to $\Omega-\mathscr{C}$ is a covering map. If $\Omega-\mathscr{C}$ is connected, then it is path connected, so $\pi$ restricted to any component of Equiv$(\Omega-\mathscr{C}, E, \tilde{E})$ is also a covering map. When $\Omega-\mathscr{C}$ is simply connected, the uniqueness of the universal covering space implies that $\pi$, restricted to any connected component, is a homeomorphism. The inverse of this restriction gives a map $\sigma$ from $\Omega-\mathscr{C}$ into Equiv$(\Omega-\mathscr{C}, E, \tilde{E})$. Then $\sigma(z)$ is a germ of an equivalence for each $z$ in $\Omega-\mathscr{C}$. Define $\Phi$: $E|_{\Omega-\mathscr{C}} \to \tilde{E}|_{\Omega-\mathscr{C}}$, an equivalence, by $[\Phi(z)]_z$ equals $\sigma(z)$.

**4.8.** If $\Omega$ and $\Omega-\mathscr{C}$ are both not simply connected then there need not be a global equivalence on $\Omega-\mathscr{C}$, as the example of flat vector bundles shows (a flat bundle is one where the curvature is identically zero):

EXAMPLE 4.8.1. Let $E$ be the trivial Hermitian line bundle over $\mathbb{C}-\{0\}$, $E=(\mathbb{C}-\{0\})\times\mathbb{C}$, with section $s\equiv e_1$ and $Ds=0$. Let $\tilde{E}$ equal $E$, with $\tilde{D}(s)=(id\theta)\,s$, where $z=re^{i\theta}$. Then the coalescing set $\mathscr{C}$ for $E$ is empty (as it is for any 1-dimensional bundle), and $E$ and $\tilde{E}$ are locally equivalent on $\mathbb{C}-\{0\}$ since the curvature is always 0. They are not globally equivalent on $\mathbb{C}-\{0\}$ since if there were $\Phi$ a global equivalence, then $\Phi(s)$ would equal $u\cdot s$ for $u$ a $C^\infty$ function on $\mathbb{C}-\{0\}$ with absolute value 1. But $\Phi$ connection-preserving implies $\tilde{D}(us)=u(Ds)$, so $du+uid\theta=0$, or $d\log u=-id\theta$ which implies $\log u=-i\theta+C$, which contradicts the multiple-valuedness of $\theta$.

Clearly the same example would work for any $\Omega$ with cohomology group $H^1(\Omega;\mathbb{R})$ non-zero. Just let $\tilde{D}(s)=i\omega s$ where $d\omega=0$ but $\omega$ is not exact.

We conjecture that the examples above exhaust all the possible types of non-global equivalence that can arise from two bundles whose curvatures and their covariant derivatives match up point-wise to infinite order.

*Conjecture* 4.9. Let $E$ and $\tilde{E}$ be Hermitian vector bundles, over $\Omega$ a manifold, with metric-preserving connections $D$ and $\tilde{D}$. Let $\mathscr{C}$ be the coalescing set of the curvature for $E$. If $E$ and $\tilde{E}$ are locally equivalent on $\Omega-\mathscr{C}$, and the following holds:

$$\Omega-\mathscr{C} \text{ is connected} \tag{4.9.1}$$

then the sheaf Equiv$(\Omega, E, \tilde{E})$, of germs of equivalences of $E$ with $\tilde{E}$, is a covering space of $\Omega$.

In particular, if $\Omega-\mathscr{C}$ is connected and $\Omega$ is simply connected, we conjecture that local equivalence on $\Omega-\mathscr{C}$ implies global equivalence on $\Omega$.

As evidence for this conjecture we have the following result:

PROPOSITION 4.10. *Let $E$ and $\tilde{E}$ be Hermitian vector bundles over $\mathbb{R}^k (k\geqslant 2)$ with metric-preserving connections $D$ and $\tilde{D}$. Let $\mathscr{C}$ be the* *coalescing set of the curvature for E. If E and $\tilde{E}$ are locally equivalent on $\mathbb{R}^k - \mathscr{C}$ and if*

$$\mathscr{C} \textit{ is contained in } X \times Y \tag{4.10.1}$$

*where X is closed and nowhere dense in $\mathbb{R}^{k-1}$ and Y is closed in $\mathbb{R}^1$ with $Y \neq \mathbb{R}^1$, then E and $\tilde{E}$ are globally equivalent.*

Note that $\mathscr{C}$ could have positive measure and that $\mathbb{R}^k - \mathscr{C}$ could be a mess topologically. Why is there monodromy nonetheless?

Note also that local equivalence off an isolated point always extends across the point:

COROLLARY 4.11. *Let E and $\tilde{E}$ be Hermitian vector bundles, over $\Omega$ a manifold, with connections D and $\tilde{D}$. If the coalescing set $\mathscr{C}$ for the curvature of E has an isolated point and E is locally equivalent to $\tilde{E}$ in a deleted neighborhood of this point, then E is equivalent to $\tilde{E}$ on a (non-deleted) neighborhood of the point.*

*Proof* (of Proposition 4.10). Let $\Theta$ be the matrix of 1-forms for $D$ relative to a global orthonormal frame for $E$; $\mathbb{R}^k$ is contractible so there is such a frame. Similarly let $\tilde{\Theta}$ be the matrix of 1-forms for $\tilde{D}$. If $U$ is to be the matrix of the global equivalence relative to the frames for $E$ and $\tilde{E}$ then we must solve

$$U\tilde{\Theta} = \Theta U + dU \tag{4.11.1}$$

on all of $\mathbb{R}^k$, for $U$ unitary and $C^\infty$.

Without loss of generality we may assume that $Y$ is $\{y \in \mathbb{R} \mid |y| \geqslant k\}$ for $k$ some positive constant. Then $\mathbb{R}^k - X \times Y$ is contractible (projection onto $\mathbb{R}^{k-1}$ is a strong deformation retract via the homotopy $(x, y)$ goes to $(x, ty)$ for $0 \leqslant t \leqslant 1$). By Proposition 4.7 there is an equivalence on all of $\mathbb{R}^k - X \times Y$, that is, there is a solution $U(x, y)$, for $x$ in $\mathbb{R}^{k-1}$ and $y$ in $\mathbb{R}^1$, of (4.11.1) on $\mathbb{R}^k - X \times Y$.

Let $A(x, y)$ be the coefficient of $dy$ in $\Theta$, so $A(x, y)$ is a $C^\infty$ $n \times n$ matrix-valued function on $\mathbb{R}^k$, and let $\tilde{A}(x, y)$ be the corresponding matrix for $\tilde{\Theta}$. Then (4.11.1) implies that $U(x, y)$ is a solution of

$$U(x, y)\, \tilde{A}(x, y) = A(x, y)\, U(x, y) + \partial U/\partial y \tag{4.11.2}$$

on $\mathbb{R}^k - X \times Y$.

By the theory of systems of linear ordinary differential equations depending on parameters, there is a matrix solution $F(x, y, Z)$ which is $C^\infty$ on *all* of $\mathbb{R}^k \times \mathbb{R}^{n^2}$ such that (4.11.2) is satisfied, namely,

$$F(x, y, z)\, \tilde{A}(x, y) = A(x, y)\, F(x, y, Z) + \partial F/\partial y \tag{4.11.3}$$

and which satisfies the initial condition $F(x, 0, Z) = Z$.

Let $\tilde{U}(x,y)$ equal $F(x,y,U(x,0))$. Then $\tilde{U}$ satisfies (4.11.2) and $\tilde{U}(x,0)=U(x,0)$. By uniqueness of solutions, we have $\tilde{U}(x,y)=U(x,y)$ on $\mathbb{R}^k - X \times Y$. Thus $\tilde{U}$ solves (4.11.1) on $\mathbb{R}^k - X \times Y$. Since $\tilde{U}$ is $C^\infty$ and $X \times Y$ is closed and nowhere-dense, continuity implies that $\tilde{U}$ satisfies (4.11.1) on all of $\mathbb{R}^k$, which is what we wanted.

4.12. *Remarks.* Of course this proof will not suffice for the conjecture, but the type of estimates used in O.D.E. do yield the Conjecture in other special cases. Furthermore, the product structure in the Proposition may be a red herring; certainly $\mathscr{C}$ could have crossings. For example, if $\mathscr{C}$ were $([-1,1] \times \{0\}) \cup (\{0\} \times [-1,1])$ in $\mathbb{R}^2$, the local equivalence on $\mathbb{R}^2 - \mathscr{C}$ would imply equivalence on $\mathbb{R}^2 - ([-1,1] \times \{0\}) \cup (\{0\} \times [-1,\infty))$ which is simply connected. Using Proposition 4.11, first on the lower half-plane and then on the right and left half-planes, would extend the equivalence to $\mathbb{R}^2 - (\{0\} \times [0,\infty))$. This can then be extended to all of $\mathbb{R}^2$.

## APPENDIX: GENERICITY

**A.1.** We conclude our discussion of connections with an appendix on genericity. We have to show that our usage of the term "generic bundle" in Definition 2.14.4 is justified. That is, the set of all connections for which $E$ is generic (in our sense) forms an open dense subset in the space of all connections on the fixed bundle $E$. Since we are primarily interested in Hermitian holomorphic bundles, we also show that the set of all Hermitian structures on a fixed holomorphic bundle, which lead to the bundle being generic with respect to the canonical connection (0.4.8), forms an open dense subset in the space of all Hermitian structures. For both of these, we use standard techniques on transversality ([GG], [H]).

The topology used in transversality is the strong (or Whitney) $C^\infty$ topology. It is a very large topology, without a countable base at any point, but does satisfy the Baire Category Theorem [H]. The general procedure is to start local and then globalize; so we may assume at first that the bundle is trivial. In this case metric-preserving connections and Hermitian structures may be identified with $C^\infty$ functions into linear spaces, so the strong $C^\infty$ topology makes sense for these objects. The strong $C^\infty$ topology can then be extended to connections (or to Hermitian structures) on any bundle $E$ over a manifold $\Omega$ by covering $\Omega$ with a locally finite covering $\{\Omega_i\}$ where $E$ is trivial over each $\Omega_i$ and proceeding as in [H, Chap. 2].

PROPOSITION A.2. (i) *Let $E$ be a $C^\infty$ n-dimensional Hermitian vector bundle over $\Omega$ a complex k-dimensional manifold. Then there exists an open dense subset $\mathscr{D}_{\mathrm{gen}}$ of the set $\mathscr{D}$ of metric-preserving connections on $E$ (in the* *strong $C^\infty$ topology), such that each metric-preserving connection $D$ in $\mathscr{D}_{\text{gen}}$ has the following property:*

*The algebra generated by the curvatures of $D$ at $z$ in $\Omega$ is*

$$\begin{cases} \textit{the full algebra } \operatorname{Hom}(E_z, E_z), \textit{ if } k \textit{ is greater than } 1; \textit{ or} \\ \qquad \textit{multiplicity free, if } k \textit{ equals } 1 \end{cases} \tag{A.2.1}$$

*for all $z$ in $\Omega - \mathscr{C}_D$, where $\mathscr{C}_D$ is a closed, nowhere-dense subset of $\Omega$. Indeed $\mathscr{C}_D$ is a union of a finite number of real sub-manifolds of $\Omega$, each sub-manifold having real co-dimension at least 1 in $\Omega$.*

(ii) *If $E$ is a holomorphic bundle then there exists an open dense subset $\mathscr{M}_{\text{gen}}$ of the set $\mathscr{M}$ of all Hermitian metrics on $E$ such that for each Hermitian metric in $\mathscr{M}_{\text{gen}}$, the corresponding canonical connection $D$* (0.4.8) *has the property above.*

*Remarks.* (1) The first part of the Proposition is of course true if $\Omega$ is any real $C^\infty$ manifold with the obvious modifications in (A.2.1) "$k$ greater than 2" and "$k$ equals 2" where $k$ is the real dimension of $\Omega$. It is just convenient to use complex notation, especially as our interest lies in the second half of the Proposition, where $\Omega$ does have to be complex.

(2) When $k$ equals 1 in Proposition A.2, there is only one curvature $\mathscr{K}^{1,1}_{11}$ which we denote by $\mathscr{K}$. The Proposition is still true if we replace "the algebra generated by the curvatures of $D$" and (A.2.1) with "the algebra generated by $\mathscr{K}$ and its first covariant derivatives at $z$ is the full algebra $\operatorname{Hom}(E_z, E_z)$." This is the direct analogue of the $k > 2$ case and may prove useful in sharpness results for the operator theory or complex curve situation.

(3) The reader who is not familiar with transversality may wonder why (A.2.1) could not be required to hold on all of $\Omega$, instead of on $\Omega - \mathscr{C}_D$. The reason is similar to the case of $C^\infty$ maps of $\mathbb{R}$ into itself. The nowhere zero maps are not dense is the space of all maps, since a map which takes on both positive and negative values cannot be approximated by maps which are never zero. The Transversality Theorem states that maps with isolated zeroes are dense.

*Proof.* Case (i). We first assume that $E$ is a trivial Hermitian bundle over $\Omega$ open in $\mathbb{C}^k$, with orthonormal frame $s$.

According to (2.15.2) and (2.15.3), a metric-preserving connection on $\Omega$ is equivalent to a $k$-tuple of $C^\infty$ $n \times n$ matrix-valued functions, namely, $\Theta_1'(s), \ldots, \Theta_k'(s)$. By (0.7.1) the curvatures are polynomials in these matrix-valued functions and their first covariant derivatives.

Let $sl(n)$ denote the set of $n \times n$ complex matrices with trace zero. Then the curvatures generate the full matrix algebra if and only if the linear maps on $sl(n)$, given by taking commutators with each of the curvatures, have joint kernel equal to 0. This is an algebraic condition on the curvatures.

To sum up: under our assumptions, the set $\mathscr{D}$ of metric-preserving connections can be identified with the set of all $C^\infty$ functions of $\Omega$ into $\mathbb{R}^{2kn^2}$ (the $k$-tuples of $n \times n$ complex matrices). Let $V$ be the direct sum of $2k+1$ copies of $\mathbb{R}^{2kn^2}$. Then the curvatures are each complex-valued polynomials on $V$ composed with the map

$$J_f = \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_k}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_k}\right)$$

of $\Omega$ into $V$, where $f : \Omega \to \mathbb{R}^{2kn^2}$ has been identified with the connection. Thus there exists a real algebraic subset $W$ of $V$ such that the curvatures generate the full matrix algebra at $z$ if and only if $J_f(z)$ is not in $W$.

If $k=1$ then $W$ equals $V$ so this does not help. But if $k$ is bigger than 1, $W$ is a proper subset of $V$. To see this, it suffices to exhibit one metric-preserving connection such that the curvatures generate everything at some point. We assume that 0 is in $\Omega$. Let $A_1, \ldots, A_k$ be self-adjoint $n \times n$ matrices and let $D$ have the matrix of connection 1-forms on $\Omega$:

$$\Theta(s) = \sum_{i=1}^{k} A_i(z_i d\bar{z}_i - \bar{z}_i dz_i) \tag{A.2.2}$$

so $\Theta(s)$ is $C^\infty$ and skew-adjoint. By (0.7.1), we have at $z=0$:

$$\begin{aligned} K(s) &= d\Theta(s) + \Theta(s) \wedge \Theta(s) \\ &= \sum_{i=1}^{k} 2A_i \, dz_i d\bar{z}_i \end{aligned} \tag{A.2.3}$$

since $\Theta(s)$ is 0 at $z=0$. Thus by (2.2.1), $\mathscr{K}_{ii}^{1,1}$ equals $A_i$ at $z=0$. But if $k>1$, then we can choose $k$ self-adjoint matrices $A_1, \ldots, A_k$ which do generate all the $n \times n$ matrices, e.g.,

$$A_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & n \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 & & & 0 \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 0 & & & 1 & 0 \end{pmatrix} \tag{A.2.4}$$

and $A_i$ arbitrary for $i>2$.

Thus $W$ is a closed, proper algebraic subset of $V$. By a Theorem of Whitney on algebraic sets [Mi], $W$ decomposes into a finite disjoint union of smooth connected sub-manifolds of $V$ (each of which is possibly non-closed, though $W$ is closed). Thus if $J_f$ is transversal to $W$, then $(J_f)^{-1}(W)$ is closed and is the union of smooth sub-manifolds of $\Omega$ each having co-dimension at least 1 since each sub-manifold in the decomposition of $W$ has codimension at least 1. For $k > 1$, let $\mathscr{D}_{\text{gen}}$ be the set of metric-preserving connections on $\Omega$ such that for the corresponding $C^\infty$ function $f$ of $\Omega$ into $\mathbb{R}^{2kn^2}$, $J_f$ is transversal to $W$. By the Transversality Theorem for jets, this is an open dense subset of all the $C^\infty$ functions of $\Omega$ into $\mathbb{R}^{2kn^2}$. If we put $\mathscr{E}_D$ equal to $(J_f)^{-1}(W)$ for each $D$ in $\mathscr{D}_{\text{gen}}$, then $\mathscr{D}_{\text{gen}}$ satisfies the Proposition.

To handle the case when $k$ equals 1, note if $\lambda_1, \ldots, \lambda_n$ are complex numbers then by use of the Vandermonde determinant we obtain:

$$\left\{\prod_{i<j} (\lambda_i - \lambda_j)\right\}^2 = \det \begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_n \\ \vdots & & \\ \lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix} \det \begin{pmatrix} 1\,\lambda_1 & \cdots & \lambda_1^{n-1} \\ \vdots & & \\ 1\,\lambda_n & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

$$= \det \begin{pmatrix} \Sigma_0 & \cdots & \Sigma_{n-1} \\ \vdots & & \\ \Sigma_{n-1} & \cdots & \Sigma_{2n-2} \end{pmatrix}$$

where $\Sigma_j = \sum_{i=1}^n \lambda_i^j$. So if $S$ is a diagonalizable $n \times n$ matrix then $S$ has distinct eigenvalues if and only if

$$\det \begin{pmatrix} \operatorname{tr} S^0 & \cdots & \operatorname{tr} S^{n-1} \\ \vdots & & \vdots \\ \operatorname{tr} S^{n-1} & \cdots & \operatorname{tr} S^{2n-2} \end{pmatrix} \tag{A.2.5}$$

is non-zero. This is an algebraic condition on $S$. Since the curvature $\mathscr{K}$ is self-adjoint when $k$ equals 1, the algebra it generates has multiplicity if and only if an algebraic condition on $\mathscr{K}$ is satisfied. Thus, just as above, there is an algebraic set $W$ contained in $V$ such that the algebra generated by the curvature at $z$ is multiplicity-free if and only if $J_f(z)$ is not in $W$. The example in (A.2.2) with $k$ equal 1 and $A_1$ as in (A.2.4) shows that $W$ is not equal to $V$. Proceeding as above, we are done in the case $k = 1$ as well, as long as $E$ is trivial over $\Omega$.

To show Remark (2), we check that a similar proof works when $k$ equals 1, if we take the algebra generated by the curvature and its first covariant derivative. We only need to show that there is one connection whose corresponding algebra at one point is everything. So let $\Theta(s)$ be defined by

$$\Theta(s) = A_1(z d\bar{z} - \bar{z} dz) + A_2(z^2 d\bar{z} - \bar{z}^2 dz) \tag{A.2.6}$$

where $A_1$ and $A_2$ are $n \times n$ and self-adjoint. Then

$$K(s) = \{2A_1 + 2(z + \bar{z}) A_2 + [A_1, A_2] |z|^2(\bar{z} - z)\} dz d\bar{z}$$

so the curvature $\mathscr{K} = \mathscr{K}_{11}^{1,1}$ satisfies

$$\mathscr{K} = 2A_1 + 2(z + \bar{z}) A_2 + O(|z|^2).$$

Since $\Theta(s)$ is zero at 0, $\mathscr{K}_z$ equals $d\mathscr{K}/dz$ at 0. Thus at 0, $\mathscr{K}$ equals $2A_1$, and $\mathscr{K}_z$ equals $2A_2$, and they generate $M(n, \mathbb{C})$.

*Case* (ii). We assume that $E$ is a trivial holomorphic bundle over $\Omega$ open in $\mathbb{C}^k$, with holomorphic frame $s$ equal $\{s_1, \ldots, s_n\}$. Then a Hermitian structure on $E$ can be identified with a $C^\infty$ map of $\Omega$ into $\mathbb{H}_n$, the space of positive definite $n \times n$ matrices. The identification is by means of the Gramian $H$, where $H$ is the matrix of inner products $((s_j, s_i))_{i,j=1}^n$. Note that $\mathbb{H}_n$ is an open subset of a linear space, the set of all $n \times n$ Hermitian matrices. Each $C^\infty$ map $H$ from $\Omega$ to $\mathbb{H}_n$ determines a unique Hermitian structure on $\Omega$.

By (0.7.3), the curvature $K$ has matrix $\bar{\partial}(H^{-1}\partial H)$ so

$$K(s) = H^{-1}\bar{\partial}\partial H - H^{-1}\bar{\partial}H\, H^{-1}\partial H. \tag{A.2.7}$$

Thus the algebra generated by the curvatures is also generated by the

$$(\det H)^{2n} \left\{ H^{-1} \frac{\partial^2 H}{\partial z_i \partial \bar{z}_j} - H^{-1} \frac{\partial H}{\partial \bar{z}_j} H^{-1} \frac{\partial H}{\partial z_i} \right\}$$

which are polynomials in the entries of $J_H^2$, where $J_H^2$ equals

$$\left( H, \frac{\partial H}{\partial z_1}, \ldots, \frac{\partial H}{\partial z_n}, \frac{\partial H}{\partial \bar{z}_1}, \ldots, \frac{\partial H}{\partial \bar{z}_n}, \frac{\partial^2 H}{\partial z_1 \partial \bar{z}_1}, \ldots, \frac{\partial^2 H}{\partial z_1 \partial \bar{z}_n}, \ldots, \frac{\partial^2 H}{\partial z_n \partial \bar{z}_n} \right).$$

Now we can use transversality just as in case (i). We only need to show that the algebraic set to which the map $J^2 H$ is transverse is of codimension at least 1. To do this we need only exhibit some $H$ for which the corresponding canonical connection is generic. So set

$$H = I + \sum |z_i|^2 A_i \tag{A.2.8}$$

where the $A_i$'s are self-adjoint. Then $H$ is positive-definite near 0. Then for $z$ equal to zero, we have by (A.2.7): $K(s)|_{z=0} = \sum A_i dz_i d\bar{z}_i$, so the algebra is generated by $A_1,..., A_k$. If we choose the $A_i$'s according to (A.2.4) we can satisfy (A.2.1) as was desired.

In order to verify Remark (2) for $k = 1$, we proceed similarly and then set

$$H = I + |z|^2 A + (z + \bar{z}) |z|^2 B \tag{A.2.9}$$

where $A$ and $B$ are self-adjoint. A computation at $z$ equal 0 shows that the curvature and its first covariant derivative generate the same algebra as $A$ and $B$.

We have proved parts (i) and (ii) of the Proposition under the assumption that $\Omega$ is in $\mathbb{C}^k$ and $E$ is trivial. That is, we have proved the local version of Proposition A.2. The proof of the global version is much more technical and we will just sketch the "patching together" process. We do this for part (i) of the Proposition. Part (ii) is similar.

First, we fix a metric-preserving connection $D_0$ for $E$ over $\Omega$. If $D$ in $\mathscr{D}$ is any metric-preserving connection, then by (0.4.2), $D - D_0$ is linear over functions and hence determines a section of $\operatorname{Hom}(E, E \otimes T^*(\Omega))$. Thus we may identify $\mathscr{D}$ with a subset of the $C^\infty$ functions from $\Omega$ into $\operatorname{Hom}(E, E \otimes T^*)$ and we give $\mathscr{D}$ the induced strong $C^\infty$ topology. Note that the identification depends on $D_0$ but the induced topology is independent of the choice of $D_0$. Furthermore, the image of $\mathscr{D}$ under this identification is closed under pointwise convergence. By Theorem 4.2 of [H], $\mathscr{D}$ is a Baire space in the strong $C^\infty$ topology, that is, a countable intersection of open dense sets in $\mathscr{D}$ is dense.

Second, we choose a countable open cover $\{\Omega_i\}$ of $\Omega$ and compact sets $L_i, L_i$ contained in $\Omega_i$, such that the cover $\{\Omega_i\}$ is locally finite, each $\Omega_i$ is a coordinate neighborhood of $\Omega$, $E$ restricted to $\Omega_i$ is trivial, with orthonormal frame $S^i$, and the interiors $L_i^0$ also cover $\Omega$.

Let $D$ be in $\mathscr{D}(\Omega_i)$, the metric-preserving connections on $\Omega_i$. Then $D$ can be identified with $f^i$, a $C^\infty$ function from $\Omega_i$ to $\mathbb{R}^{2kn^2}$ as in case (i) above. We define $\mathscr{D}_{\text{gen}}(\Omega_i; L_i)$ by

$$\mathscr{D}_{\text{gen}}(\Omega_i; L_i) = \{D \in \mathscr{D}(\Omega_i) | J_{f^i} \text{ is transverse to } W \text{ on } L_i\} \tag{A.2.10}$$

where $W$ is the algebraic set defined in case (i) above. There we showed, in effect, that $\mathscr{D}_{\text{gen}}(\Omega_i; L_i)$ is open and dense in $\mathscr{D}(\Omega_i)$.

We define $\mathscr{D}_{\text{gen}}(\Omega; L_i)$ to be the set of $D$ in $\mathscr{D}$ such that $D|_{\Omega_i}$ is in $\mathscr{D}_{\text{gen}}(\Omega_i; L_i)$ and define $\mathscr{D}_{\text{gen}}(\Omega)$ by

$$\mathscr{D}_{\text{gen}}(\Omega) = \bigcap \mathscr{D}_{\text{gen}}(\Omega; L_i). \tag{A.2.11}$$

The $\mathscr{D}_{\text{gen}}(\Omega, L_i)$ are trivially open in $\mathscr{D}$. By local finiteness of the $L_i$ and by definition of the strong $C^\infty$ topology [H], $\mathscr{D}_{\text{gen}}(\Omega)$ is open in $\mathscr{D}$.

The $\mathscr{D}_{\text{gen}}(\Omega; L_i)$ are dense in $\mathscr{D}$ by the usual globalization in the Transversality Theorem using "bump functions." This construction keeps us within the set $\mathscr{D}$ as is easily checked. Thus by the Baire property for $\mathscr{D}$, $\mathscr{D}_{\text{gen}}(\Omega)$ is dense in $\mathscr{D}$.

For $D$ in $\mathscr{D}_{\text{gen}}(\Omega)$, let $\mathscr{C}_D$ be the set

$$\mathscr{C}_D = \bigcup_i (J_{fi})^{-1}(W). \tag{A.2.12}$$

By definition of $W$, $D$ satisfies (A.2.1) on $\Omega - \mathscr{C}_D$, and, furthermore, $\Omega_i \cap (J_{fj})^{-1}(W)$ is contained in $(J_{fi})^{-1}(W)$. Thus $L_i^0 \cap \mathscr{C}_D$ is $(J_{fi}|_{L^0})^{-1}(W)$, which is a union of a finite number of closed, real sub-manifolds of $L_i^0$, each having codimension at least 1. Since the $L_i^0$ cover $\Omega$, this completes the proof of the Proposition.

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